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Optimality and duality for nonsmooth semi-infinite E-convex multi-objective programming with support functions

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Abstract. In this paper, we study a nonsmooth semi-infinite multi-objective E-convex programming problem involving support functions. We derive sufficient optimality conditions for the primal problem. We formulate Mond-Weir type dual for the primal problem and establish weak and strong duality theorems under various generalized E-convexity assumptions.

Keywords: Nonsmooth semi-infinite multi-objective optimization / generalized e-convexity / duality

1 Introduction

Semi-infinite multi-objective programming consider several conflicting objective functions have to be optimized over a feasible set described by infinite number of inequality constraints. Semi-infinite programming problems have occupied the attention of a number of mathematicians due to their applications in many areas such as in engineering, robotics, and transportation problems, see [1]. Optimality conditions and duality results for semi-infinite programming problems have been studied see, [2–10]. Optimality and duality results for semi-infinite multi-objective programming problems that involved differentiable functions were obtained by Caristi et al. [11]. Several kinds of constraints qualifications were defined by Kanzi and Nobakhtian [12] and they obtained necessary and sufficient optimality conditions for nonsmooth semi-infinite multi-objective programming problems. Mishra et al. [13] proved necessary and sufficient optimality conditions for nondifferential semi-infinite programming problems involving square root of quadratic functions, for more details see [14]. Mond and Schechter [15] have constructed symmetric duality of both Wolfe and Mond-Weir types for nonlinear programming problems where the objective contains the support function. Optimality and duality for a nondifferentiable nonlinear programming problem involving support function have been obtained by Husain et al. [16], see for more details [17–20]. In other hand, convexity and their generalizations play an important role in optimization theory. Youness point of view of convexity is based on the effecting of an operator E on the domain on which functions are defined [21,22]. This kind of convexity is called E-convexity and can be viewed in many fields such as in differential geometry when a manifold is deformed by an operator E. In the field of physical chemistry an E-convexity can be occurred when the binding force f between elements construct a crystal effect by a solution E. In mathematical programming, the notion of E-convexity of functions plays an important role in solving the problem of type composite model problem [23] such as the problem

$$\min||F|| s.t. x \in M,\ \text{where}\ f = ||\ ||, E(x) = F(x)\ \text{and}\ (f \circ E)x = ||F(x)||.$$

This paper is organized as follows: In Section 2, we mention some definitions and preliminaries. In Section 3, the sufficient optimality conditions for multi-objective semi-infinite E-convex programming problems involving support functions are established. In Section 4, we formulate Mond-Weir type dual for multi-objective semi-infinite E-convex programming problems involving support functions and establish weak, strong and strict-converse duality theorems under generalized E-convexity assumptions.

2 Definitions and preliminaries

In this section, we present some definitions and results, which will be needed in this article. Let R^n be the n-dimensional Euclidean space and R^n_+ be the nonnegative orthant of R^n. Let $\langle\, .\,,.\rangle$ denotes the Euclidean inner product and $||\,.\,||$ be Euclidean norm in R^n. Given a nonempty

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set $D\subseteq R^n$, we denote the closure of D by $\overline{D}$ and convex cone (containing origin) by $cone(D)$. The native polar cone and the strictly negative polar cone are defined respective by

$$D^{\leq} := \{d\in R^n | \langle x, d\rangle \leq 0,\ \forall x\in D\},$$

$$D^{<} := \{d\in R^n | \langle x, d\rangle < 0,\ \forall x\in D\}.$$

Definition 1 *[24] Let $D\subseteq R^n$. The contingent cone T(D, x) at $\overline{x}\in\overline{D}$ is defined by*

$$T\left(D, \overline{x}\right) := \{d\in R^n | \exists t_k \downarrow 0, \exists d_k \to d : \overline{x} + t_k d_k \in D, \forall k\in N\}.$$

Definition 2 *[24] A function $f : R^n \to R$ is said to be Lipschitz near $x\in R^n$, if there exist a positive constant K and a neighborhood N of x such that for any y, z ∈ N, we have*

$$|f(y) - f(z)| \leq K||y - z||.$$

The function f is said to be locally Lipschitz on R^n if it is Lipschitz near x for every $x\in R^n$.

Definition 3 *[24] The Clarke generalized directional derivative of a locally Lipschitz function f at $x\in R^n$ in the direction $d\in R^n$, denoted by $f^{\circ}(x, d)$, is defined as*

$$f^{\circ}(x, d) = \lim_{t\uparrow 0, y\to x} \frac{sup(f(y + td) - f(y))}{t},$$

where $y\in R^n$.

Definition 4 *[24] The Clarke generalized subdifferential of f at $x\in R^n$ is denoted by $\partial cf(x)$, defined as*

$$\partial cf(x) = \{\xi\in R^n : f^{\circ}(x, d) \geq \langle \xi, d\rangle, \forall d\in R^n\}.$$

Definition 5 *[21] A set $M\subseteq R^n$ is said to be E-convex set with respect to an operator $E : R^n \to R^n$ if $\lambda E(x) + (1-\lambda)E(y)\in M$ for each x, y ∈ M and $0\leq\lambda\leq 1$.*

Every E-convex set with respect to an operator $E: R^n\to R^n$ is a convex set when $E = I$. If M_1 *and* M_2 are E-convex sets, then $M_1 \cap M_2$ is E-convex set but $M_1 \cup M_2$ is not necessarily E-convex set. If $E: R^n \to R^n$ is a linear map, and $M_1, M_2 \subseteq R^n$ are E-convex sets, then $M_1 + M_2$ is E-convex set.

Example 1 *Let $E : R^2 \to R^2$ be defined as $E(x, y) = (0, y)$. The set $M = \{(x, y)\in R^2 : (x, y) = \lambda_1(0, 0) + \lambda_2(1, 2) + \lambda_3(0, 3)\} \cup \{(x, y)\in R^2 : (x, y) = \lambda_1(0, 0) + \lambda_2(0, -3) + \lambda_3(-1, -2)\}, \lambda_i \geq 0, \sum_{i=1}^{3}\lambda_i = 1\}$ is an E-convex set with respect to the operator E.*

Definition 6 *A locally Lipschitz function $f : R^n \to R$ is said to be E-convex with respect to an operator $E : R^n \to R^n$ at $x^*\in R^n$ if*

$$(f\circ E)x - (f\circ E)x^* \geq \langle \xi, Ex - Ex^*\rangle$$

for each $x\in R^n$ and every $\xi\in\partial cf(Ex^*)$.

The function f is said to be E-convex near $x^*\in R^n$ if it is E-convex at each point of neighborhood of $x^*\in R^n$.

Definition 7 *A locally Lipschitz function $f : R^n \to R$ is said to be strictly E-convex with respect to an operator $E : R^n \to R^n$ at $x^*\in R^n$ if*

$$(f\circ E)x - (f\circ E)x^* > \langle \xi, Ex - Ex^*\rangle$$

for each $x\in R^n$, $x\neq x^*$ and every $\xi\in\partial cf(Ex^*)$.

The function f is said to be strictly E-convex near $x^*\in R^n$ if it is strictly E-convex at each point of neighborhood of $x^*\in R^n$.

Proposition 1 *[21] If $g_i : R^n \to R$, $i = 1, 2, \ldots, m$ is E-convex with respect to $E : R^n \to R^n$ then the set $M = \{x\in R^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$ is E-convex set.*

Definition 8 *A locally Lipschitz function $f : R^n \to R$ is said to be pseudo E-convex with respect to $E : R^n \to R^n$ at $x^*\in R^n$ if*

$$\langle \xi, Ex - Ex^*\rangle \geq 0,\ \text{for some } \xi\in\partial cf(Ex^*) \Rightarrow (f\circ E)x \geq (f\circ E)x^*,$$

for each $x\in R^n$ and every $\xi\in\partial cf(Ex^*)$.

Definition 9 *A locally Lipschitz function $f : R^n \to R$ is said to be strictly pseudo E-convex with respect to $E : R^n \to R^n$ at $x^*\in R^n$ if*

$$\langle \xi, Ex - Ex^*\rangle \geq 0, \text{for some } \xi\in\partial cf(Ex^*) \Rightarrow (f\circ E)x > (f\circ E)x^*,$$

for each $x\in R^n$, $x\neq x^*$ and every $\xi\in\partial cf(Ex^*)$.

Definition 10 *A locally Lipschitz function $f : R^n \to R$ is said to be quasi E-convex with respect to $E : R^n \to R^n$ at $x^*\in R^n$ if*

$$(f\circ E)x \leq (f\circ E)x^* \Rightarrow \langle \xi, Ex - Ex^*\rangle \leq 0,$$

for each $x\in R^n$ and every $\xi\in\partial cf(Ex^*)$.

The function f is said to be quasi E-convex near $x^*\in R^n$ if it is quasi E-convex at each point of neighborhood of $x^*\in R^n$.

Proposition 2 *[21] If $g_i : R^n \to R$, $i = 1, 2, \ldots, m$ is quasi E-convex with respect to $E : R^n \to R^n$ then the set $M = \{x\in R^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$ is E-convex set.*

Remark 1

- *Every E-convex function is also quasi E-convex with respect to same $E : R^n \to R^n$, but not conversely.*
- *Every E-convex function is also pseudo E-convex with respect to same $E : R^n \to R^n$, but not conversely.*
- *Every strictly E-convex function is also strictly pseudo E-convex with respect to same $E : R^n \to R^n$, but not conversely.*

Let C be a nonempty compact E-convex set in R^n. The support function $S(.|C) : R^n \to R\cup\{+\infty\}$ is given by

$$S(x|C) = \max\{\langle x, Ez\rangle : z\in C\}.$$

Example 2 *Let $E : R^2 \to R^2$ be defined as $E(y_1, y_2) = (0, y_2)$. If $C = \{(y_1, y_2)\in R\ 2 : (y_1, y_2) = \lambda_1\ (0, 0) + \lambda_2(1\ , 2) + \lambda_3(0, 3)\} \cup \{(y_1, y_2)\in R^2 : (y_1, y_2) = \lambda_1(0, 0) + \lambda_2(0, -3) + \lambda_3(-1, -2)\}, \lambda_i \geq 0, \sum_{i=1}^{3}\lambda_i = 1\}$, then the support function*

$S(./C): R \to R \cup \{+\infty\}$ *is given by*

$$S(x|C) = \max\{\langle x, E(y_1, y_2)\rangle : (y_1, y_2) \in C\} \\ = \max\langle x, (0, y_2)\rangle,$$

i.e.

$$S(x|C) = 3|x|.$$

The support function, being convex and everywhere finite, has a Clark subdifferential [24], in the sense of convex analysis. Its subdifferential is given by

$$\partial S(x|C) = \{z \in C : \langle x, Ez\rangle = S(x|C)\}.$$

In this paper, we consider the following nonsmooth semi-infinite multi-objective E-convex programming problem:

$$(P) \qquad \min \ f_j(x) + S(x|C_j) \ , \qquad j = 1, \ldots, p,$$

subject to

$$g_i(x) \le 0, \quad i \in I,$$

$$x \in R^n.$$

where I is an index set which is possibly infinite, $f_j(x)$, $j = 1, \ldots, p$ and $g_i(x)$, $i \in I$ are locally Lipschitz E-convex functions from R^n to $R \cup \{+\infty\}$. Let M denote the E-convex feasible set of (P).

$$M := \{x \in R^n | g_i(x) \le 0, \forall i \in I\}$$

Let $x^* \in M$. We denote $I(x^*) = \{i \in I | (g_i \circ E)x^* = 0\}$, the index set of active constraints and let

$$F(Ex^*) := \bigcup_{j=1}^{p} \partial_c(f_j(Ex^*) + S(Ex^*|C_j))$$

$$G(Ex^*) := \bigcup_{i \in I(x^*)} \partial_c g_i(Ex^*).$$

The following constraint qualifications are generalization of constraint qualifications from [12] for multi-objective E-convex programming problem with support functions (P).

Definition 11 *We say that:*

- The Abedie constraint qualification (ACQ) holds at $\tilde{x} \in M$ if $G^{\le}(\tilde{x}) \subseteq T(M, \tilde{x})$.
- The Basic constraint qualification (BCQ) holds at $\tilde{x} \in M$ if $T^{\le}(M, \tilde{x}) \subseteq cone(G(x))$.
- The Regular constraint qualification (RCQ) holds at $x \in M$ if $F^{<}(\tilde{x}) \cap G^{\le}(\tilde{x}) \subseteq T(M, \tilde{x})$.

Definition 12 *A feasible point* $x^* \in M$ *is said to be weakly efficient solution for (P) if there is no* $x \in M$ *such that*

$$f_j(x) + S(x|C_j) < f_j(x^*) + S(x^*|C_j), \quad \text{for all } j = 1, \ldots, p.$$

3 Optimality conditions

In this section, we prove the sufficient optimality conditions for considered nonsmooth semi-infinite multi-objective E-convex programming problem (P) as follows:

Theorem 1 (Necessary optimality conditions) *Let* $E: R^n \to R^n$ *and* x^* *be a feasible solution of (P). Assume that* Ex^* *be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at* $E(x^*)$. *If cone(G (*Ex^**)) is closed, then there exist* $\tau_j \ge 0$, $z_j \in C_j$ (for $j = 1, 2, \ldots, p$) and $\lambda_i \ge 0$ (for $i \in I(x^*)$) with $\lambda_i \ne 0$ *for finitely many indices i, such that*

$$0 \in \sum_{j=1}^{p} \tau_j[\partial_c f_j(Ex^*) + z_j] + \sum_{I(x^*)} \lambda_i \partial_c g_i(Ex^*), \tag{1}$$

$$\sum_{j=1}^{p} \tau_j = 1, \tag{2}$$

$$\langle z_j, Ex^*\rangle = S(x^*|C_j), j = 1, 2, \ldots, p. \tag{3}$$

Proof: See Theorem 3.4 (ii) of Kanzi and Nobakhtian [12].

Theorem 2 (Sufficient optimality conditions) *Let* $E: R^n \to R^n$ *and* x^* *be a feasible solution of (P). Assume that there exist* $\tau_j \ge 0$, $z_j \in C_j$ (for $j = 1, 2, \ldots, p$) and $\lambda_i \ge 0$ (for $i \in I(x^*)$) with $\lambda_i \ne 0$ *for finitely many indices i, such that necessary optimality conditions (1)–(3) hold at* x^*. *If* $\tau_j(f_j(.) + \langle z_j, .\rangle)$ *for* $j = 1, 2, \ldots, p$ *are pseudo E-convex at* x^* *and* $\lambda_i g_i(.)$, $i \in I(x^*)$ *are quasi E-convex at* x^* *with respect to the same* E *and* $f_j(Ex) \le f_j(x)$, $j = 1, \ldots, p$, $\forall x \in M$. *Then,* Ex^* *is a weakly efficient solution for (P).*

Proof: Suppose, contrary to the result, that $Ex^* \in M$, is not a weakly efficient solution for (P). Then, there exists a feasible point $x \in M$ for (P) such that

$$f_j(x) + S(x|C_j) < f_j(Ex^*) + S(Ex^*|C_j) \ , \quad \text{for all } j = 1, \ldots, p,$$

but $f_j(Ex) \le f_j(x)$ and $\tau_j \ge 0$, *for* $j = 1, 2, \ldots, p$, so we have

$$\sum_{j=1}^{p} \tau_j[f_j(Ex) + S(x|C_j)] < \sum_{j=1}^{p} \tau_j[f_j(Ex^*) + S(Ex^*|C_j)]. \tag{4}$$

Since $\langle z_j, Ex\rangle \le S(x|C_j)$, $j = 1, 2, \ldots, p$ and the assumption $\langle z_j, Ex^*\rangle = S(x^*|C_j)$, $j = 1, 2, \ldots, p$, we have

$$\sum_{j=1}^{p} \tau_j[f_j(Ex) + \langle z_j, Ex\rangle] < \sum_{j=1}^{p} \tau_j[f_j(Ex^*) + \langle z_j, Ex^*\rangle]. \tag{5}$$

Now, from equation (1), there exist $\xi_j \in \partial cf_j(Ex^*)$ and $\zeta_i \in \partial cg_i(Ex^*)$ such that

$$\sum_{j=1}^{p} \tau_j(\xi_j + z_j) + \sum_{I(x^*)} \lambda_i \zeta_i = 0. \tag{6}$$

Since Ex is a feasible point for (P) where M is E-convex set and $\lambda_i g_i\,(Ex^*) = 0,\ i \in I(x^*)$, we have

$$\sum_{I(x^*)} \lambda_i g_i(Ex) \ \leq\ \sum_{I(x^*)} \lambda_i g_i(Ex^*), \tag{7}$$

and from quasi E-convexity of $g_i,\ i \in I(x^*)$, we get

$$\left\langle Ex - Ex^*, \sum_{I(x^*)} \lambda_i \zeta_i \right\rangle \ \leq\ 0,$$

by using (6), we have

$$\left\langle Ex - Ex^*, \sum_{j=1}^{p} \tau_j(\xi_j + z_j) \right\rangle \geq 0.$$

Thus, from pseudo E-convexity of $\tau_j(f_j(.) + \langle\, z_j,\, .\rangle)$, *for* $j = 1,\ 2,\ ...,\ p$, we get

$$\sum_{j=1}^{p} \tau_j[f_j(Ex) + \langle z_j, Ex\rangle] \geq \sum_{j=1}^{p} \tau_j[f_j(Ex^*) + \langle z_j, Ex^*\rangle],$$

which contradicts (5). Thus Ex^* is a weakly efficient solution for (P).

The following corollary is a direct consequence of Remark 1 and Theorem 2.

Corollary 1 *Let* $\mathrm{E} : \mathrm{R}^n \to \mathrm{R}^n$ *and* x^* *be a feasible solution of (P). Assume that there exist* $\tau_j \geq 0$, $z_j \in C_j$(for $j = 1, 2, ..., p$) and $\lambda_i \geq 0$(for $i \in I(x^*)$) with $\lambda_i \neq 0$ *for finitely many indices i, such that necessary optimality conditions (1)–(3) hold at* x^*. *If* $\tau_j(f_j(.) + \langle\, z_j,\, .\rangle)$ *for* $j = 1, 2, ..., p$ *and* $\lambda_i g_i(.),\ \in I(x^*)$ *are E-convex at* x^* *with respect to the same* E *and* $f_j(Ex) \leq f_j(x)$, $j = 1, ..., p$, $\forall x \in M$. *Then,* Ex^* *is a weakly efficient solution for (P).*

Example 3 *Let* $\mathrm{E} : \mathrm{R}^2 \to \mathrm{R}^2$ *be defined as* $E(x_1, x_2) = (\frac{x_1}{2}, x_2)$ *and let M be given by*

$$M = \{(x_1, x_2) \in R^2 : x_1 + 2x_2 - 6 \ \leq\ 0, x_1 - 4x_2 \ \leq\ 0, x_2 \geq 0, x_1 \geq 0\}.$$

Consider the bicriteria E-convex programming problem

$$(P)\min\,(f_1(x) + S(x|C_1), f_2(x) + S(x|C_2))$$

$$\text{Subjectto } x \in M,$$

where $f_1(x_1, x_2) = x_1^3$ and $f_2(x_1,\ x_2) = (x_2 - x_1)^3$ and $S(x|C_1) = S(x|C_2) = \frac{1}{4}|x_2|$ where $x = (x_1,\ x_2)$*for* $C_1 = C_2 = \{(0,\ x_2) : -12 \leq x_2 \leq 0\}$. It is clear that M is E-convex with respect to E and

$E(M) = \{\,(x_1, x_2) \in R^2 : x_1 + x_2 - 3 \leq 0, x_1 - 2x_2 \leq 0, x_2 \geq 0, x_1 \geq 0\,\}$.

By choosing $(g_1 \circ E)x^* = x_1^* - 2x_2^*$ as the active constraint of (P) then $I\,(x^*) = 1$. It is clear that all defined functions are locally Lipschitz functions at Ex^* and $\partial f_1(Ex^*) = (12\alpha^2,\ 0)$, $\partial f_2(Ex^*) = (-3\alpha^2,\ 3\alpha^2)$, $\partial g_1(Ex^*) = (1, -2)$ where $\alpha \in [0,\ 1]$. Since $\tau_j\,(f_j(x) + \langle\, z_j,\, x\rangle)$*for* $j = 1, 2$ are pseudo E-convex and $\lambda_1 g_1(x)$ are quasi E-convex at x^* with respect to same E and conditions (1)–(3) of theorem (1) holds at $x^* \in M$ as there exist $\tau_1 = \tau_2 = \lambda_= \frac{1}{2}, z_1 = 0$, $z_2 = -\alpha(9\alpha + 2, 3\alpha + 1), \xi_1 = (12\alpha^2, 0), \xi_2 = (-3\,\alpha^2, 3\,\alpha^2)$, $\zeta_1 = (2\alpha, \alpha)$, where $\alpha \in [0,\ 1]$ such that

$$\sum_{j=1}^{2} \tau_j(\xi_j + z_j) + \sum_{I(x^*)} \lambda_i \zeta_i = 0.$$

Then there is no $x \in M$ such that

$$f_j(x) + S(x|C_j) < f_j(Ex^*) + S(Ex^*|C_j), j = 1, 2,$$

and hence $Ex^* = (x_1^*, x_2^*)$ where $x_1^* = 2x_2^*$ and $x_2^* \in [0, 1]$ are weakly efficient solutions for (P).

4 Duality criteria

Many authors have formulated Mond-Weir type dual and established duality results in various optimization problems with support functions; see [10,15,17,20,21,25] and the references therein. Following the above mentioned works, we formulate Mond-Weir type dual for nonsmooth semi-infinite E-convex programming problem with support function (P) and establish duality theorems.

$$\max\big(f_1(Ey) + \langle z_1, Ey\rangle, \ldots, f_p(Ey) + \langle z_p, Ey\rangle\big), \tag{D}$$

subject to

$$0 \in \sum_{j=1}^{p} \tau_j[\partial_c f_j(Ey) + z_j] + \sum_{i \in I} \lambda_i \partial_c g_i(Ey), \tag{8}$$

$$\sum_{i \in I} \lambda_i g_i(Ey) \geq 0. \tag{9}$$

We now discuss the weak, strong and strict converse duality for (P) and (D).

Theorem 3 (Weak Duality) *Let x be feasible for (P) and* $(y,\ \tau,\ \lambda,\ z_1,\ ...,\ z_p)$ *be feasible for (D). If* $\tau_j(f_j(.) + \langle\, z_j,\ .\rangle)$ *for* $j = 1,\ 2,\ ...,\ p$ *are pseudo E-convex at* y *and* $\lambda_i g_i(.),\ i \in I$ *are quasi E-convex at* y *with respect to the same* E *and* $f_j(Ex) \leq f_j(x)$, $j = 1, ..., p$, $\forall x \in M$. *Then the following cannot hold:*

$$f_j(x) + S\big(x|C_j\big) < f_j(Ey) + \langle z_j, Ey\rangle\ , \quad \text{for all } j = 1, \ldots, p.$$

Proof: Let x be feasible for (P) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for (D), then from (8), there exist $\xi_j \in \partial\, cf_j(Ey)$ and $\zeta_i \in \partial\, cg_i(Ey)$ such that

$$\sum_{j=1}^{p} \tau_j\big(\xi_j + z_j\big) + \sum_{i \in I} \lambda_i \zeta_i = 0. \tag{10}$$

We proceed to the result of the theorem by contradiction. Assume that

$$f_j(x) + S\big(x|C_j\big) < f_j(Ey) + \langle z_j, Ey\rangle,\ \text{ forall } j = 1, \ldots, p.$$

But $f_j\ (Ex) \leq f_j(x)$ and $\tau_j \geq 0$, for $j = 1,\ 2,\ ...,\ p$, so we have

$$\sum_{j=1}^{p}\tau_j[f_j(Ex) + S(x|C_j)] < \sum_{j=1}^{p}\tau_j[f_j(Ey) + \langle z_j, Ey\rangle], \tag{11}$$

and by using the inequality $\langle z,\ Ex\rangle \leq S(x|C)$, we get

$$\sum_{j=1}^{p}\tau_j[f_j(Ex) + \langle z_j, Ex\rangle] < \sum_{j=1}^{p}\tau_j[f_j(Ey) + \langle z_j, Ey\rangle]. \tag{12}$$

Now, since Ex is feasible for (P) where M is E-convex set and $(y,\ \tau,\ \lambda,\ z_1,\ \ldots,\ z_p)$ is feasible for (D), we have

$$\sum_{i\in I}\lambda_i g_i(Ex)\ \leq\ 0\ \leq\ \sum_{i\in I}\lambda_i g_i(Ey),$$

and from definition of quasi E-convexity of $g_i(x)$, $i \in I$ *at* y, we have

$$\left\langle Ex - Ey, \sum_{i\in I}\lambda_i\xi_j\right\rangle \geq 0, \tag{13}$$

for each $x \in M$ and every and $\zeta_i \in \partial\, cg_i\ (Ex)$. From (10) in (13), we get

$$\left\langle Ex - Ey, \sum_{j=1}^{p}\tau_j(\xi_j + z_j)\right\rangle \geq 0,$$

for each $x \in M$ and some $\xi_j \in \partial\, cf_j\ (Ey)$. Thus, from the definition of pseudo E-convexity of $\tau_j\ (f_j(.) + \langle\ z_j, .\rangle)$ for $j = 1,\ 2,\ ...,\ p$, we have

$$\sum_{j=1}^{p}\tau_j[f_j(Ex) + \langle z_j, Ex\rangle] \geq \sum_{j=1}^{p}\tau_j[f_j(Ey) + \langle z_j, Ey\rangle],$$

which contradicts (12). Hence,

$$f_j(x) + S\left(x|C_j\right) < f_j(Ey) + \langle z_j, Ey\rangle, \forall j = 1, \ldots, p,$$

cannot hold.

The following corollary is a direct consequence of Remark 1 and Theorem 3.

Corollary 2 *Let x be feasible for (P) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for (D). If $\tau_j(f_j(.) + \langle z_j,\ .\rangle)$ for $j = 1,\ 2,\ \ldots,\ p$ are E-convex at y and $\lambda_i g_i(.)$, $i \in I$ are E-convex at y with respect to the same E and $f_j(Ex) \leq f_j(x)$, $j = 1, \ldots, p$, $\forall\, x \in M$. Then the following cannot hold:*

$$f_j(x) + S(x|C_j) < f_j(Ey) + \langle z_j, Ey\rangle, \forall j = 1, \ldots, p.$$

The following example shows that the generalized B-invexity imposed in the above theorem is essential.

Example 4 *[26] We consider the following problem:*

$$(P)\min\left(f_1(x) + S(x|C_1), f_2(x) + S(x|C_2)\right)$$

Subject to

$$g_i(x)\ \leq\ 0, i\in I$$

$$x\in R,$$

where $f_1(x) = -2x$, $f_2(x) = x^2$, $S(x|C_1) = S(x|C_2) = |x|$ for $C_1 = C_2 = [-1,\ 1]$ and $g_i(x) = -i|x|$, for $i \in I := N$. It is clear that the feasible set of (P) is M := R and for y = 1 ∈ M, I(y) = I. Let us formulate Mond-Weir dual of (P) as follow:

$$(D)\max\{(f_1\circ E)y + z_1, (f_2\circ E)y + z_2\}$$

Subject to

$$g_i(Ey)\ \leq\ 0, i\in I$$

$$0\in\sum_{j=1}^{2}\tau_j[\partial f_j(Ey) + z_j] + \sum_{i\in I}\lambda_i\partial g_i(Ey),$$

$$\sum_{i\in I}\lambda_i g_i(Ey) \geq 0,$$

where $y\in R, \tau_j \geq 0, \sum_{j=1}^{2}\tau_j = 1, \lambda_i \geq 0$ with $\lambda = (\lambda_i)_{i\in I} \neq 0$ for finitely many indices $i \in N$ and $z_j \in C_j$ for $j = 1,\ 2$. By choosing $y^* = 0, \tau_1 = \tau_2 = \frac{1}{2}, \lambda = (1,0,0,\ldots), z_1 = 1, z_2 = 0$. We have $(y,\ \tau,\ \lambda,\ z_1,\ z_2)$ be feasible for (D).

Note that $\lambda_i g_i(.)$ is not quasi E-convex at y with respect to $E(y) = y$ and that $f_1(x) + S(x|C_1) = -1 < f_1(Ey) + \langle\ z_1, Ey\rangle = 0$ holds. This means that pseudo E-convexity and quasi E-convexity assumptions are essential for weak duality.

The following theorem gives strong duality relation between the primal problem (P) and the dual problem (D).

Theorem 4 (Strong Duality) *Let $E: R^n \to R^n$ and x^* be a feasible solution of (P). Assume that Ex^* be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at x^* and* cone *$(G(x^*))$ is closed. If the pseudo E-convexity and quasi E-convexity assumptions of the weak duality theorem are satisfied, and $f_j(Ex) \leq f_j(x)$, $j = 1, \ldots, p$, $\forall\, x \in M$. Then there exists $(\tau,\ \lambda,\ z_1,\ \ldots,\ z_p)$ such that $(x^*,\ \tau,\ \lambda,\ z_1,\ \ldots,\ z_p)$ is a weakly efficient solution for (D) and the respective objective values are equal.*

Proof: Since Ex^* is a weakly efficient solution for (P) at which the suitable constraints qualification holds and *cone* $(G(x^*))$ is closed, from the Kuhn-Tucker necessary conditions, there exists $(\tau,\ \lambda,\ z_1,\ \ldots,\ z_p)$ such that $(x^*,\ \tau,\ \lambda,\ z_1,\ \ldots,\ z_p)$ is feasible for (D).

From weak duality theorem (3), the following cannot hold:

$$f_j(x) + S(x|C_j) < f_j(Ex^*) + \langle z_j, Ex^*\rangle, \text{for } j = 1, \ldots, p.$$

Since $\langle z,\ Ex\rangle \leq S(x|C)$, and $f_j\ (Ex) \leq f_j\ (x)$, $j = 1,\ \ldots,\ p$, we have

$$f_j(Ex) + \langle z_j, Ex\rangle < f_j(Ex^*) + \langle z_j, Ex^*\rangle, \text{for } j = 1, \ldots, p,$$

cannot hold, and hence $(x^*, \tau, \lambda, z_1, \ldots, z_p)$ is a weakly efficient solution for (D) and the objective values of (P) and (D) are equal at x.

The following corollary is a direct consequence of Remark 1 and Theorem 4.

Corollary 3 *Let* $E: R^n \to R^n$ *and* x^* *be a feasible solution of (P). Assume that* Ex^* *be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at* x^* *and* $\mathrm{cone}(G(x^*))$ *is closed. If the E-convexity assumptions of the weak duality theorem are satisfied, and* $f_j(Ex) \le f_j(x)$, $j = 1, \ldots, p$, $\forall x \in M$. *Then there exists* $(\tau, \lambda, z_1, \ldots, z_p)$ *such that* $(x^*, \tau, \lambda, z_1, \ldots, z_p)$ *is a weakly efficient solution for (D) and the respective objective values are equal.*

The following theorem gives strict converse duality relation between the primal problem (P) and the dual problem (D).

Theorem 5 (Strict converse duality) *Let* Ex^* *be a weakly efficient solution for (P) at which a suitable constraints qualification from Definition 11 holds at* x^* *and* $\mathrm{cone}(G(x^*))$ *is closed. Let* $\tau_j(f_j(.) + \langle z_j, . \rangle)$ *for j = 1, 2, ..., p be pseudo E-convex and* $\lambda_i g_i(.)$, $i \in I$ *be quasi E-convex with respect to the same* E. *If* $(\tilde{x}, \tau, \lambda, z_1, \ldots, z_p)$ *is a weak efficient solution for (D) and* $\tau_j(f_j(.) + \langle z_j, . \rangle)$ *for j = 1, 2, ..., p are strictly pseudo E-convex at* $\tilde{x}$, *then* $\tilde{x} = x^*$

Proof: We prove the result of theorem by contradiction. Assume that $\tilde{x} \neq x^*$. Then by strong duality Theorem (4) there exists $(\tau, \lambda, z_1, \ldots, z_p)$ such that $(Ex^*, \tau, \lambda, z_1, \ldots, z_p)$ is a weakly efficient solution for (P) and the inequality

$$f_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle \le f_j(Ex^*) + \langle z_j, Ex^* \rangle, \text{for} \quad j = 1, \ldots, p,$$

cannot be hold. i.e.

$$\sum_{j=1}^{p} f_j(Ex^*) + \langle z_j, Ex^* \rangle < \sum_{j=1}^{p} f_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle. \quad (14)$$

Now, since Ex^* is a weakly efficient solution for (P), $\lambda_i \ge 0$ and $(\tilde{x}, \tau, \lambda, z_1, \ldots, z_p)$ is a weakly efficient solution for (D), we have

$$\sum_{i \in I} \lambda_i g_i(Ex^*) \le \sum_{i \in I} \lambda_i g_i(E\tilde{x}),$$

and from the definition of quasi E-convexity of $\lambda_i g_i(.)$, $i \in I$

$$\left\langle Ex^* - E\tilde{x}, \sum_{i \in I} \lambda_i \zeta_i \right\rangle \le 0, \quad (15)$$

for every $x^* \in M$ and every $\zeta_i \in \partial c g_i(E\tilde{x})$. By substituting from (10) in (15), we get

$$\left\langle Ex^* - E\tilde{x}, \sum_{j=1}^{p} \tau_j(\xi_j + z_j) \right\rangle \ge 0.$$

for each $x^* \in M$ and some $\xi_j \in \partial c f_j(E\tilde{x})$. Thus from strict pseudo E-convexity of $\tau_j(f_j(.) + \langle z_j, . \rangle)$ for $j = 1, 2, \ldots, p$ at $\tilde{x}$, we get

$$\sum_{j=1}^{p} f_j(Ex^*) + \langle z_j, Ex^* \rangle > \sum_{j=1}^{p} f_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle. \quad (16)$$

which contradicts (1). Therefore, $\tilde{x} = x^*$.

Some Applications

Let us briefly review a few interesting applications. Nonsmooth semi-infinite multi-objective programming problems very naturally lend to a highly disaggregated formulation. Computation of economic equilibria is a very promising area of application for nonsmooth semi-infinite multi-objective programming problems. A paper [27] give example evidence of the solving-power of ACCPM (analytical center cutting plane method) on these reputedly difficult problems. At the end, we would like to mention applications to nonsmooth semi-infinite multi-objective programming problems. In the first application [28], ACCPM (analytical center cutting plane method) is used to solve a Lagrangian relaxation of the capacitate multi-item lot sizing problem with set-up times. A full integration of ACCPM in a column generation, or Lagrangian relaxation, framework, for structured nonsmooth semi-infinite multi-objective programming problems, shows that the reliability and robustness of ACCPM in applications where a non-differentiable problem must be solved repeatedly makes it a very powerful alternative to sub gradient optimization [29].

5 Conclusions

This paper investigates the optimality conditions and duality for nonsmooth semi-infinite E-convex multi-objective programming with support functions. The obtained results extended and improved corresponding results of [26,21] to nonsmooth E-convex case. By applying the obtained results, one can study fractional programming, set-valued optimization and variational inequalities and so on. For instance, we can apply the obtained Kuhn-Tucker necessary and sufficient conditions to study the optimality conditions and duality for nonsmooth multiobjective programming problems with generalized E-convexity. We can also apply the Kuhn-Tucker sufficient conditions to consider the solvability of some vector variational inequalities.

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