Optimality and duality for nonsmooth semi-infinite E-convex multi-objective programming with support functions

Tarek Emam¹,²,*

¹ Department of Mathematics, Faculty of Science, Jouf University, P.O. Box 2014, Sakaka, Saudi Arabia
² Department of Mathematics, Faculty of Science, Suez University, P.O. Box 43533, Suez, Egypt

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Abstract. In this paper, we study a nonsmooth semi-infinite multi-objective E-convex programming problem involving support functions. We derive sufficient optimality conditions for the primal problem. We formulate Mond-Weir type dual for the primal problem and establish weak and strong duality theorems under various generalized E-convexity assumptions.

Keywords: Nonsmooth semi-infinite multi-objective optimization / generalized e-convexity / duality

1 Introduction

Semi-infinite multi-objective programming consider several conflicting objective functions have to be optimized over a feasible set described by infinite number of inequality constraints. Semi-infinite programming problems have occupied the attention of a number of mathematicians due to their applications in many areas such as in engineering, robotics, and transportation problems, see [1]. Optimality conditions and duality results for semi-infinite programming problems have been studied see, [2–10]. Optimality and duality results for semi-infinite multi-objective programming problems that involved differentiable functions were obtained by Caristi et al. [11]. Several kinds of constraints qualifications were defined by Kanzi and Nobakhtian [12] and they obtained necessary and sufficient optimality conditions for nonsmooth semi-infinite multi-objective programming problems. Mishra et al. [13] proved necessary and sufficient optimality conditions for nondifferential semi-infinite programming problems involving square root of quadratic functions, for more details see [14]. Mond and Schechter [15] have constructed symmetric duality of both Wolfe and Mond-Weir types for nonlinear programming problems where the objective contains the support function. Optimality and duality for a nondifferentiable nonlinear programming problem involving support function have been obtained by Husain et al. [16], see for more details [17–20]. In other hand, convexity and their generalizations play an important role in optimization theory. Youness point of view of convexity is based on the effecting of an operator E on the domain on which functions are defined [21,22]. This kind of convexity is called E-convexity and can be viewed in many fields such as in differential geometry when a manifold is deformed by an operator E. In the field of physical chemistry an E-convexity can be occurred when the binding force f between elements construct a crystal effect by a solution E. In mathematical programming, the notion of E-convexity of functions plays an important role in solving the problem of type composite model problem [23] such as the problem

\[
\begin{align*}
\text{min} & \ |F|/s.t.x \in M, \text{ where } f = ||.||, E(x) \\
& = F(x) \text{ and } (f \circ E)x = |F(x)|.
\end{align*}
\]

This paper is organized as follows: In Section 2, we mention some definitions and preliminaries. In Section 3, the sufficient optimality conditions for multi-objective semi-infinite E-convex programming problems involving support functions are established. In Section 4, we formulate Mond-Weir type dual for multi-objective semi-infinite E-convex programming problems involving support functions and establish weak, strong and strict-converse duality theorems under generalized E-convexity assumptions.

2 Definitions and preliminaries

In this section, we present some definitions and results, which will be needed in this article. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be the nonnegative orthant of $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $||.||$ be Euclidean norm in $\mathbb{R}^n$. Given a nonempty
set $D \subseteq R^n$, we denote the closure of $D$ by $\overline{D}$ and convex cone (containing origin) by cone($D$). The native positive and convex cone are defined respectively by

$$D^+ := \{d \in R^n | (x, d) \leq 0, \forall x \in D\},$$

$$D^- := \{d \in R^n | (x, d) > 0, \forall x \in D\}.$$

**Definition 1** [24] Let $D \subseteq R^n$. The contingent cone $T(D, x)$ at $x \in D$ is defined by

$$T(D, x) := \{d \in R^n | \exists t_k \geq 0, \exists \delta_k \rightarrow d : x + t_k \delta_k \in D, \forall k \in N\}.$$

**Definition 2** [24] A function $f: R^n \rightarrow R$ is said to be Lipschitz near $x \in R^n$, if there exist a positive constant $K$ and a neighborhood $N$ of $x$ such that for any $y, z \in N$, we have

$$|f(y) - f(z)| \leq K||y - z||.$$

The function $f$ is said to be locally Lipschitz on $R^n$ if it is Lipschitz near $x$ for every $x \in R^n$.

**Definition 3** [24] The Clarke generalized directional derivative of a locally Lipschitz function $f$ at $x \in R^n$ in the direction $d \in R^n$, denoted by $f'(x, d)$, is defined as

$$f'(x, d) = \lim_{t \downarrow 0, y \rightarrow x} \frac{sup(f(y + td) - f(y))}{t},$$

where $y \in R^n$.

**Definition 4** [24] The Clarke generalized subdifferent of $f$ at $x \in R^n$ is denoted by $\partial cf(x)$, defined as

$$\partial cf(x) = \{\xi \in R^n : f'(x, d) \geq (\xi, d), \forall d \in R^n\}.$$

**Definition 5** [21] A set $M \subseteq R^n$ is said to be $E$-convex with respect to an operator $E: R^n \rightarrow R^n$ if $\lambda E(x) + (1 - \lambda)E(y) \in M$ for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

Every $E$-convex set with respect to an operator $E: R^n \rightarrow R^n$ is a convex set when $E = I$. If $M_1$ and $M_2$ are $E$-convex sets, then $M_1 \cap M_2$ is $E$-convex set but $M_1 \cup M_2$ is not necessarily $E$-convex set. If $E: R^n \rightarrow R^n$ is a linear map, then $M_1, M_2 \subseteq R^n$ are $E$-convex sets, then $M_1 + M_2$ is $E$-convex set.

**Example 1** Let $E: R^2 \rightarrow R^2$ be defined as $E(x, y) = (0, y)$. The set $M = \{(x, y) \in R^2 : (x, y) = \lambda_1(0, 0) + \lambda_2(1, 2) + \lambda_3(0, 3) \}$ is an $E$-convex set with respect to operator $E$.

**Definition 6** A locally Lipschitz function $f: R^n \rightarrow R$ at $x^* \in R^n$ is said to be $E$-convex with respect to an operator $E: R^n \rightarrow R^n$ at $x^* \in R^n$ if

$$(f \circ E)x - (f \circ E)x^* \geq \langle \xi, E(x) - E(x^*) \rangle$$

for each $x \in R^n$ and every $\xi \in \partial cf(E(x^*))$.

The function $f$ is said to be $E$-convex near $x^* \in R^n$ if it is $E$-convex at each point of neighborhood of $x^* \in R^n$.

**Definition 7** A locally Lipschitz function $f: R^n \rightarrow R$ is said to be strictly $E$-convex with respect to an operator $E: R^n \rightarrow R^n$ at $x^* \in R^n$ if

$$f(\xi) - (f \circ E)x^* > \langle \xi, E(x) - E(x^*) \rangle$$

for each $x \in R^n$, $x \neq x^*$ and every $\xi \in \partial cf(E(x^*))$.

The function $f$ is said to be strictly $E$-convex near $x^* \in R^n$ if it is strictly $E$-convex at each point of neighborhood of $x^* \in R^n$.

**Proposition 1** [21] If $g_i: R^n \rightarrow R$, $i = 1, 2, ..., m$ is $E$-convex with respect to $E: R^n \rightarrow R^n$ then the set $M = \{x \in R^n : g_i(x) \leq 0, i = 1, 2, ..., m\}$ is $E$-convex set.

**Definition 8** A locally Lipschitz function $f: R^n \rightarrow R$ is said to be pseudo $E$-convex with respect to $E: R^n \rightarrow R^n$ at $x^* \in R^n$ if

$$\langle \xi, E(x) - E(x^*) \rangle \geq 0, \text{ for some } \xi \in \partial cf(E(x^*)) \Rightarrow (f \circ E)x \geq (f \circ E)x^*$$

for each $x \in R^n$ and every $\xi \in \partial cf(E(x^*))$.

**Definition 9** A locally Lipschitz function $f: R^n \rightarrow R$ is said to be strictly pseudo $E$-convex with respect to $E: R^n \rightarrow R^n$ at $x^* \in R^n$ if

$$\langle \xi, E(x) - E(x^*) \rangle > 0, \text{ for some } \xi \in \partial cf(E(x^*)) \Rightarrow (f \circ E)x > (f \circ E)x^*$$

for each $x \in R^n$, $x \neq x^*$ and every $\xi \in \partial cf(E(x^*))$.

**Definition 10** A locally Lipschitz function $f: R^n \rightarrow R$ is said to be quasi $E$-convex with respect to $E: R^n \rightarrow R^n$ at $x^* \in R^n$ if

$$f(\xi) - (f \circ E)x^* \Rightarrow \langle \xi, E(x) - E(x^*) \rangle \leq 0$$

for each $x \in R^n$ and every $\xi \in \partial cf(E(x^*))$.

The function $f$ is said to be quasi $E$-convex near $x^* \in R^n$ if it is quasi $E$-convex at each point of neighborhood of $x^* \in R^n$.

**Proposition 2** [21] If $g_i: R^n \rightarrow R$, $i = 1, 2, ..., m$ is quasi $E$-convex with respect to $E: R^n \rightarrow R^n$ then the set $M = \{x \in R^n : g_i(x) \leq 0, i = 1, 2, ..., m\}$ is $E$-convex set.

**Remark 1**

- Every $E$-convex function is also quasi $E$-convex with respect to same $E: R^n \rightarrow R^n$, but not conversely.
- Every $E$-convex function is also pseudo $E$-convex with respect to same $E: R^n \rightarrow R^n$, but not conversely.
- Every strictly $E$-convex function is also strictly pseudo $E$-convex with respect to same $E: R^n \rightarrow R^n$, but not conversely.

Let $C$ be a nonempty compact $E$-convex set in $R^n$. The support function $S(C): R^n \rightarrow R \cup \{+ \infty\}$ is given by

$$S(x) = max \{\langle x, E(x) \rangle : x \in C\}.$$
functions from $\mathbb{R}^n$ to $\mathbb{R}$ is given by

$$S(x|C) = \max \{ x, E(y_1, y_2) : (y_1, y_2) \in C \} = \max \{ x, (0, y_2) \},$$

i.e.

$$S(x|C) = 3|x|.$$

The support function, being convex and everywhere finite, has a Clark subdifferential [24], in the sense of convex analysis. Its subdifferential is given by

$$\partial S(x|C) = \{ z \in C : \langle x, Ez \rangle = S(x|C) \}.$$

In this paper, we consider the following nonsmooth semi-infinite multi-objective E-convex programming problem:

$$(P) \quad \min \ f_j(x) + S(x|C_j), \quad j = 1, \ldots, p,$$

subject to

$$g_i(x) \leq 0, \quad i \in I,$$

$$x \in \mathbb{R}^n.$$ where $I$ is an index set which is possibly infinite, $f_j(x)$, $j = 1, \ldots, p$ and $g_i(x)$, $i \in I$ are locally Lipschitz E-convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{ + \infty \}$. Let $M$ denote the E-convex feasible set of (P).

$$M := \{ x \in \mathbb{R}^n | g_i(x) \leq 0, \forall i \in I \}$$

Let $x^* \in M$. We denote $I(x^*) = \{ i \in I : (g_i \circ E)x^* = 0 \}$, the index set of active constraints and let

$$F(Ex^*) := \bigcup_{j=1}^p \partial (f_j(Ex^*) + S(Ex^*|C_j))$$

$$G(Ex^*) := \bigcup_{i \in I(x^*)} \partial g_i(Ex^*).$$

The following constraint qualifications are generalization of constraint qualifications from [12] for multi-objective E-convex programming problem with support functions (P).

**Definition 11** We say that:
- The Abedie constraint qualification (ACQ) holds at $\tilde{x} \in M$ if $G^*(\tilde{x}) \subseteq T(M, \tilde{x})$.
- The Basic constraint qualification (BCQ) holds at $\tilde{x} \in M$ if $T^*(\tilde{x}) \subseteq \text{cone}(G(\tilde{x}))$.
- The Regular constraint qualification (RCQ) holds at $x \in M$ if $F^*(\tilde{x}) \cap G^*(\tilde{x}) \subseteq T(M, \tilde{x})$.

**Definition 12** A feasible point $x^* \in M$ is said to be weakly efficient solution for (P) if there is no $x \in M$ such that

$$f_j(x) + S(x|C_j) < f_j(x^*) + S(x^*|C_j), \quad \text{for all } j = 1, \ldots, p.$$  

### 3 Optimality conditions

In this section, we prove the sufficient optimality conditions for considered nonsmooth semi-infinite multi-objective E-convex programming problem (P) as follows:

**Theorem 1** (Necessary optimality conditions) Let $E : \mathbb{R}^n \to \mathbb{R}^m$ and $x^*$ be a feasible solution of (P). Assume that $Ex^*$ be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at $E(x^*)$. If $\text{cone}(G(Ex^*))$ is closed, then there exist $\tau_j \geq 0$, $z_j \in C_j$ (for $j = 1, 2, \ldots, p$) and $\lambda_i \geq 0$ (for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices $i$, such that

$$0 \leq \sum_{j=1}^p \tau_j [\partial f_j(Ex^*)] + z_j + \sum_{i \in I(x^*)} \lambda_i \partial g_i(Ex^*), \quad (1)$$

$$\sum_{j=1}^p \tau_j = 1, \quad (2)$$

$$\langle z_j, Ex^* \rangle = S(x^*|C_j), j = 1, 2, \ldots, p. \quad (3)$$

**Proof:** See Theorem 3.4 (ii) of Kanzi and Nobakhtian [12].

**Theorem 2** (Sufficient optimality conditions) Let $E : \mathbb{R}^n \to \mathbb{R}^m$ and $x^*$ be a feasible solution of (P). Assume that there exist $\tau_j \geq 0$, $z_j \in C_j$ (for $j = 1, 2, \ldots, p$) and $\lambda_i \geq 0$ (for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices $i$, such that necessary optimality conditions (1)–(3) hold at $x^*$. If $\tau_j f_j(\cdot) + (z_j, \cdot)$ for $j = 1, 2, \ldots, p$ are pseudo E-convex at $x^*$ and $\lambda_i g_i(\cdot)$, $i \in I(x^*)$ are quasi E-convex at $x^*$ with respect to the same $E$ and $E(Ex^*) \leq f_j(x)$, $j = 1, 2, \ldots, p$, $\forall \ x \in M$. Then, $Ex^*$ is a weakly efficient solution for (P).

**Proof:** Suppose, contrary to the result, that $Ex^* \in M$, is not a weakly efficient solution for (P). Then, there exists a feasible point $x \in M$ for (P) such that

$$f_j(x) + S(x|C_j) < f_j(Ex^*) + S(Ex^*|C_j), \quad \text{for all } j = 1, \ldots, p,$$

but $f_j(Ex) \leq f_j(x)$ and $\tau_j \geq 0$, for $j = 1, 2, \ldots, p$, so we have

$$\sum_{j=1}^p \tau_j [f_j(Ex) + S(x|C_j)] < \sum_{j=1}^p \tau_j [f_j(Ex^*) + S(Ex^*|C_j)]. \quad (4)$$

Since $\langle z_j, Ex \rangle \leq S(x|C_j), j = 1, 2, \ldots, p$ and the assumption $\langle z_j, Ex^* \rangle = S(x^*|C_j), j = 1, 2, \ldots, p$, we have

$$\sum_{j=1}^p \tau_j [f_j(Ex) + \langle z_j, Ex \rangle] < \sum_{j=1}^p \tau_j [f_j(Ex^*) + \langle z_j, Ex^* \rangle]. \quad (5)$$

Now, from equation (1), there exist $\xi_j \in \partial c f_j (Ex^*)$ and $\zeta_i \in \partial c g_i (Ex^*)$ such that

$$\sum_{j=1}^p \tau_j (\xi_j + z_j) + \sum_{i \in I(x^*)} \lambda_i \zeta_i = 0. \quad (6)$$
Since $Ex$ is a feasible point for (P) where $M$ is E-convex set and $\lambda, g_i (Ex) = 0, \ i \in I(x^*)$, we have

$$\sum_{i \in I(x^*)} \lambda_i g_i (Ex) \leq \sum_{i \in I(x^*)} \lambda_i g_i (Ex^*)$$

(7)

and from quasi E-convexity of $g_i, \ i \in I(x^*)$, we get

$$\langle Ex - Ex^*, \sum_{i \in I(x^*)} \lambda_i \xi_i \rangle \leq 0,$n

by using (6), we have

$$\langle Ex - Ex^*, \sum_{j=1}^{p} \tau_j (\xi_j + z_j) \rangle \geq 0.$$

Thus, from pseudo E-convexity of $\tau_j (f_j (\cdot) + \langle z_j, \cdot \rangle)$, for $j = 1, 2, \ldots, p$, we get

$$\sum_{j=1}^{p} \tau_j (f_j (Ex) + \langle z_j, Ex \rangle) \geq \sum_{j=1}^{p} \tau_j (f_j (Ex^*) + \langle z_j, Ex^* \rangle),$$n

which contradicts (5). Thus $Ex^*$ is a weakly efficient solution for (P).

The following corollary is a direct consequence of Remark 1 and Theorem 2.

**Corollary 1** Let $E : R^n \rightarrow R^n$ and $x^*$ be a feasible solution of (P). Assume that there exist $\tau_j \geq 0, z_j \in C_j$ (for $j = 1, 2, \ldots, p$) and $\lambda_i \geq 0$ (for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices $i$, such that necessary optimality conditions (1)–(3) hold at $x^*$. If $\tau_j (f_j (\cdot) + \langle z_j, \cdot \rangle)$ for $j = 1, 2, \ldots, p$ and $\lambda_i g_i (\cdot), \ i \in I(x^*)$ are E-convex at $x^*$ with respect to the same $E$ and $f_j (Ex) \leq f_j (x_j), \ j = 1, \ldots, p, \ \forall x \in M$. Then, $Ex^*$ is a weakly efficient solution for (P).

**Example 3** Let $E : R^2 \rightarrow R^2$ be defined as $E(x_1, x_2) = (\frac{1}{2}, x_2)$ and let $M$ be given by

$$M = \{(x_1, x_2) \in R^2 : x_1 + 2x_2 - 6 \leq 0, x_1 - 4x_2 \leq 0, x_2 \geq 0, x_1 \geq 0\}.$$

Consider the bicriteria E-convex programming problem

$$(P) \min (f_1 (x) + S(x|C_1), f_2 (x) + S(x|C_2))$$

Subject to $x \in M,$

where $f_1 (x_1, x_2) = x_1^2$ and $f_2 (x_1, x_2) = (x_2 - x_1)^2$ and $S(x|C_1) = S(x|C_2) = \frac{1}{2} |x_2| \ \text{where} \ x = (x_1, x_2)$ for $C_1 = C_2 = \{(0, x_2) : -2 \leq x_2 \leq 0\}$. It is clear that $M$ is E-convex with respect to $E$ and $E(M) = \{(x_1, x_2) \in R^2 : x_1 + x_2 - 3 \leq 0, x_1 - 2x_2 \leq 0, x_2 \geq 0, x_1 \geq 0\}.$

By choosing $(g \circ E)x^* = x_1^* - 2x_2^*$ as the active constraint of (P) then $I (x^*) = 1$. It is clear that all defined functions are locally Lipschitz functions at $Ex^*$ and $\partial f_1 (Ex^*) = (12x_1^*, 0), \ \partial f_2 (Ex^*) = (-3x_2^*, 3x_1^*), \ \partial g_i (Ex^*) = (1, -2)$ where $\alpha \in [0, 1]$. Since $\tau_j (f_j (\cdot) + \langle z_j, x \rangle)$ for $j = 1, 2$ are pseudo E-convex and $\lambda_i g_i (\cdot)$ are quasi E-convex at $x^*$ with respect to same $E$ and conditions (1)–(3) of theorem (1) holds at $x^* \in M$ as there exist $\tau_1 = \tau_2 = \lambda = \frac{1}{2}, z_1 = 0, z_2 = -\alpha (9x_2 + 2\alpha, 3\alpha + 1), \xi_1 = (12x_1^*, 0), \xi_2 = (-3x_2^*, 3x_1^*), \ \xi_1 = (2\alpha, \alpha)$, where $\alpha \in [0, 1]$ such that

$$\sum_{j=1}^{2} \tau_j (\xi_j + z_j) + \sum_{i \in I} \lambda_i \xi_i = 0.$$

Then there is no $x \in M$ such that

$$f_j (x) + S(x|C_j) < f_j (Ex^*) + S(Ex^*|C_j), \ j = 1, 2,$$

and hence $Ex^* = (x_1^*, x_2^*)$ where $x_1^* = 2x_2^*$ and $x_2^* \in [0, 1]$ are weakly efficient solutions for (P).

**4 Duality criteria**

Many authors have formulated Mond-Weir type dual and established duality results in various optimization problems with support functions; see [10,15,17,20,21,25] and the references therein. Following the above mentioned works, we formulate Mond-Weir type dual for nonsmooth semi-infinite E-convex programming problem with support function (P) and establish duality theorems.

$$\max (f_1 (Ey) + \langle z_1, Ey \rangle, \ldots, f_p (Ey) + \langle z_p, Ey \rangle), \ \text{(D)}$$

subject to

$$0 \leq \sum_{j=1}^{p} \tau_j \partial f_j (Ey) + z_j + \sum_{i \in I} \lambda_i \partial g_i (Ey),$$

$$\sum_{i \in I} \lambda_i g_i (Ey) \geq 0.$$n

We now discuss the weak, strong and strict converse duality for (P) and (D).

**Theorem 3 (Weak Duality)** Let $x$ be feasible for (P) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for (D). If $\tau_j (f_j (\cdot) + \langle z_j, \cdot \rangle)$ for $j = 1, 2, \ldots, p$ and $\lambda_i g_i (\cdot), \ i \in I$ are E-convex at $y$ with respect to the same $E$ and $f_j (Ex) \leq f_j (x_j), \ j = 1, \ldots, p, \ \forall x \in M.$ Then the following cannot hold:

$$f_j (x) + S(x|C_j) < f_j (Ey) + \langle z_j, Ey \rangle, \ \text{for all} \ j = 1, \ldots, p.$$n

**Proof:** Let $x$ be feasible for (P) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for (D), then from (8), there exist $\xi \in \partial cf_j (Ey)$ and $\xi_i \in \partial cg_i (Ey)$ such that

$$\sum_{j=1}^{p} \tau_j (\xi_j + z_j) + \sum_{i \in I} \lambda_i \xi_i = 0.$$n

We proceed to the result of the theorem by contradiction. Assume that

$$f_j (x) + S(x|C_j) < f_j (Ey) + \langle z_j, Ey \rangle, \ \text{for all} \ j = 1, \ldots, p.$$n


But \( f_j(Ex) \leq f_j(x) \) and \( \tau_j \geq 0 \), for \( j = 1, 2, \ldots, p \), so we have
\[
\sum_{j=1}^{p} \tau_j[f_j(Ex) + S(x|C_j)] < \sum_{j=1}^{p} \tau_j[f_j(Ey) + \langle z_j, Ey \rangle],
\]
and by using the inequality \( \langle z, Ex \rangle \leq S(x|C) \), we get
\[
\sum_{j=1}^{p} \tau_j[f_j(Ex) + \langle z_j, Ex \rangle] < \sum_{j=1}^{p} \tau_j[f_j(Ey) + \langle z_j, Ey \rangle].
\]
(11)

Now, since \( Ex \) is feasible for \( (P) \) where \( M \) is E-convex set and \( (y, \tau, \lambda, z_1, \ldots, z_p) \) is feasible for \( (D) \), we have
\[
\sum_{i \in I} \lambda_i g_i(Ex) \leq 0 \leq \sum_{i \in I} \lambda_i g_i(Ey),
\]
and from definition of quasi E-convexity of \( g_i(x) \), \( i \in I \) at \( y \), we have
\[
\langle Ex - Ey, \sum_{i \in I} \lambda_i \xi_i \rangle \geq 0,
\]
for each \( x \in M \) and every \( \xi_i \in \partial g_i(Ex) \). From (10) in (13), we get
\[
\langle Ex - Ey, \sum_{j=1}^{p} \tau_j(\xi_j + z_j) \rangle \geq 0,
\]
for each \( x \in M \) and some \( \xi_j \in \partial c f_j(Ex) \). Thus, from the definition of pseudo E-convexity of \( \tau_j(f_j(\cdot) + \langle z_j, \cdot \rangle) \) for \( j = 1, 2, \ldots, p \), we have
\[
\sum_{j=1}^{p} \tau_j[f_j(Ex) + \langle z_j, Ex \rangle] \geq \sum_{j=1}^{p} \tau_j[f_j(Ey) + \langle z_j, Ey \rangle],
\]
which contradicts (12). Hence,
\[
f_j(x) + S(x|C_j) < f_j(Ey) + \langle z_j, Ey \rangle, \forall j = 1, \ldots, p,
\]
cannot hold.

The following corollary is a direct consequence of Remark 1 and Theorem 3.

**Corollary 2** Let \( x \) be feasible for \( (P) \) and \( (y, \tau, \lambda, z_1, \ldots, z_p) \) be feasible for \( (D) \). If \( \tau_j(\xi_j + \langle z_j, \cdot \rangle) \) for \( j = 1, 2, \ldots, p \) are E-convex at \( y \) and \( \lambda g_i(\cdot) \), \( i \in I \) are E-convex at \( y \) with respect to the same \( E \) and \( f_i(Ex) \leq f_i(x) \), \( j = 1, \ldots, p, \forall x \in M \). Then there follows cannot hold:
\[
f_j(x) + S(x|C_j) < f_j(Ey) + \langle z_j, Ey \rangle, \forall j = 1, \ldots, p.
\]

The following example shows that the generalized B-invexity imposed in the above theorem is essential.

**Example 4** [26] We consider the following problem:
\[
(P) \text{ min } (f_1(x) + S(x|C_1), f_2(x) + S(x|C_2))
\]
Subject to
\[
g_i(x) \leq 0, i \in I
\]
\( x \in R \),
where \( f_1(x) = -2x \), \( f_2(x) = x^2 \), \( S(x|C_1) = S(x|C_2) = |x| \) for \( C_1 = C_2 = [-1, 1] \) and \( g_1(x) = -|x| \), \( i \in I = N \). It is clear that the feasible set of \( (P) \) is \( M = \mathbb{R} \) and for \( y = 1 \in \mathbb{M} \), \( I(y) = 1 \). Let us formulate Mond-Weir dual of \( (P) \) as follow:
\[
(D) \text{ max } \{ (f_1 \circ E)y + z_1, (f_2 \circ E)y + z_2 \}
\]
Subject to
\[
g_i(Ey) \leq 0, i \in I
\]
\( 0 \in \sum_{j=1}^{2} \tau_j \partial f_j(Ey) + z_j + \sum_{i \in I} \lambda_i \partial g_i(Ey), \)
\( \lambda \sum_{i \in I} \lambda_i g_i(Ey) \geq 0, \)
where \( y \in \mathbb{E}, \tau_j \geq 0, \sum_{j=1}^{2} \tau_j = 1, \lambda_i \geq 0 \) with \( \lambda = (\lambda_i)_{i \in I} \neq 0 \) for finitely many indices \( i \in N \) and \( z_i \in C_1 \) for \( j = 1, 2 \). By choosing \( y^i = 0, \tau_1 = \tau_2 = \frac{1}{2}, \lambda = (1, 0, 0, \ldots), z_1 = 1, z_2 = 0 \). We have \( (y^i, \tau, \lambda, z_1, z_2) \) be feasible for \( (D) \).

Note that \( \lambda g_i(\cdot) \) is not quasi E-convex at \( y \) with respect to \( E(y) = y \) and that \( f_1(x) + S(x|C_1) = -1 < f_1(Ey) + \langle z_1, Ey \rangle = 0 \) holds. This means that pseudo E-convexity and quasi E-convexity assumptions are essential for weak duality.

The following theorem gives strong duality relation between the primal problem \( (P) \) and the dual problem \( (D) \).

**Theorem 4 (Strong Duality)** Let \( E : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( x^* \) be a feasible solution of \( (P) \). Assume that \( Ex^* \) be a weakly efficient solution of \( (P) \) and a suitable constraints qualification from Definition (11) holds at \( x^* \) and \( cone(G(x^*)) \) is closed. If the pseudo E-convexity and quasi E-convexity assumptions of the weak duality theorem are satisfied, and \( f_i(Ex) \leq f_i(x), j = 1, \ldots, p, \forall x \in M \). Then there exists \( (\tau, \lambda, z_1, \ldots, z_p) \) such that \( (x^*, \tau, \lambda, z_1, \ldots, z_p) \) is a weakly efficient solution for \( (D) \) and the respective objective values are equal.

**Proof:** Since \( Ex^* \) is a weakly efficient solution for \( (P) \) at which the suitable constraints qualification holds and \( cone(G(x^*)) \) is closed, from the Kuhn-Tucker necessary conditions, there exists \( (\tau, \lambda, z_1, \ldots, z_p) \) such that \( (x^*, \tau, \lambda, z_1, \ldots, z_p) \) is feasible for \( (D) \).

From weak duality theorem (3), the following cannot hold:
\[
f_j(x) + S(x|C_j) < f_j(Ex^*) + \langle z_j, Ex^* \rangle, \forall j = 1, \ldots, p.
\]
Since \( \langle z, Ex \rangle \leq S(x|C) \), and \( f_j(Ex) \leq f_j(x), j = 1, \ldots, p \), we have
\[
f_j(Ex) + \langle z_j, Ex \rangle < f_j(Ex^*) + \langle z_j, Ex^* \rangle, \forall j = 1, \ldots, p.
\]
cannot hold, and hence \((x^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((P)\) and the objective values of \((P)\) and \((D)\) are equal at \(x\).

The following corollary is a direct consequence of Remark 1 and Theorem 4.

**Corollary 3** Let \(E: \mathbb{R}^n \to \mathbb{R}^n\) and \(x^*\) be a feasible solution of \((P)\). Assume that \(Ex^*\) be a weakly efficient solution of \((P)\) and a suitable constraints qualification from Definition 11 holds at \(x^*\) and \(\text{cone}(G(x^*))\) is closed. If the \(E\)-convexity assumptions of the weak duality theorem are satisfied, and \(f_j(Ex^*) \leq f_j(x), j = 1, \ldots, p, \forall x \in M.\) Then there exists \((\tau, \lambda, z_1, \ldots, z_p)\) such that \((x^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((D)\) and the respective objective values are equal.

The following theorem gives strict converse duality relation between the primal problem \((P)\) and the dual problem \((D)\).

**Theorem 5 (Strict converse duality)** Let \(Ex^*\) be a weakly efficient solution for \((P)\) at which a suitable constraints qualification from Definition 11 holds at \(x^*\) and \(\text{cone}(G(x^*))\) is closed. Let \(\tau_j(f_j(.) + (z_j .))\) for \(j = 1, 2, \ldots, p\) be pseudo \(E\)-convex and \(\lambda_i g_i(.), i \in I\) be quasi \(E\)-convex with respect to the same \(E\). If \((\hat{x}, \tau, \lambda, z_1, \ldots, z_p)\) is a weak efficient solution for \((D)\) and \(\tau_j(f_j(.) + (z_j .))\) for \(j = 1, 2, \ldots, p\) are strictly pseudo \(E\)-convex at \(\hat{x}\), then \(\hat{x} = x^*\).

**Proof:** We prove the result of theorem by contradiction. Assume that \(\hat{x} \neq x^*\). Then by strong duality Theorem (4) there exists \((\tau, \lambda, z_1, \ldots, z_p)\) such that \((Ex^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((P)\) and the inequality

\[
f_j(Ex^*) + (z_j, Ex^*) \leq f_j(Ex^*) + (z_j, Ex^*),
\]

for \(j = 1, \ldots, p,\)
cannot be hold. i.e.

\[
\sum_{j=1}^{p} f_j(Ex^*) + (z_j, Ex^*) < \sum_{j=1}^{p} f_j(Ex) + (z_j, Ex) \tag{14}
\]

Now, since \(Ex^*\) is a weakly efficient solution for \((P)\), \(\lambda_i \geq 0\) and \((\hat{x}, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((D)\), we have

\[
\sum_{i \in I} \lambda_i g_i(Ex^*) \leq \sum_{i \in I} \lambda_i g_i(Ex),
\]

and from the definition of quasi \(E\)-convexity of \(\lambda_i g_i(.), i \in I\)

\[
\left< Ex^* - Ex, \sum_{i \in I} \lambda_i \xi_i \right> \leq 0, \tag{15}
\]

for every \(x^* \in M\) and every \(\xi \in \partial cg_i(Ex)\). By substituting from (10) in (15), we get

\[
\left< Ex^* - Ex, \sum_{j=1}^{p} \tau_j(\xi_j + z_j) \right> \geq 0.
\]

for each \(x^* \in M\) and some \(\xi \in \partial cg_i(Ex)\). Thus from strict pseudo \(E\)-convexity of \(\tau_j(f_j(.) + (z_j .))\) for \(j = 1, 2, \ldots, p\) at \(\hat{x}\), we get

\[
\sum_{j=1}^{p} f_j(Ex^*) + (z_j, Ex^*) > \sum_{j=1}^{p} f_j(Ex) + (z_j, Ex) \tag{16}
\]

which contradicts (1). Therefore, \(\hat{x} = x^*\).

**Some Applications**

Let us briefly review a few interesting applications. Nonsmooth semi-infinite multi-objective programming problems very naturally lend to a highly disaggregated formulation. Computation of economic equilibria is a very promising area of application for nonsmooth semi-infinite multi-objective programming problems. A paper [27] give example evidence of the solving-power of ACCPM (analytical center cutting plane method) on these reputedly difficult problems. At the end, we would like to mention applications to nonsmooth semi-infinite multi-objective programming problems. In the first application [28], ACCPM (analytical center cutting plane method) is used to solve a Lagrangian relaxation of the capacitate multi-item lot sizing problem with set-up times. A full integration of ACCPM in a column generation, or Lagrangian relaxation, framework, for structured nonsmooth semi-infinite multi-objective programming problems, shows that the reliability and robustness of ACCPM in applications where a non-differentiable problem must be solved repeatedly makes it a very powerful alternative to sub gradient optimization [29].

5 Conclusions

This paper investigates the optimality conditions and duality for nonsmooth semi-infinite \(E\)-convex multi-objective programming with support functions. The obtained results extended and improved corresponding results of [26,21] to nonsmooth \(E\)-convex case. By applying the obtained results, one can study fractional programming, set-valued optimization and variational inequalities and so on. For instance, we can apply the obtained Kuhn-Tucker necessary and sufficient conditions to study the optimality conditions and duality for nonsmooth multiobjective programming problems with generalized \(E\)-convexity. We can also apply the Kuhn-Tucker sufficient conditions to consider the solvability of some vector variational inequalities.

**References**


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