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Study of solutions to a class of certain parabolic systems with variable exponents

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Abstract – In this paper, the authors study an initial and boundary value problem to a system of evolution $p(x)$-Laplacian systems coupled with general nonlinear terms:

$$a_i(x)u_{it} - \operatorname{div}\left(|\nabla u_i|^{p_i(x)-2}\nabla u_i\right) = f_i(x, u_1, u_2), \quad (i = 1,\, 2).$$

The authors translate the parabolic equation into the elliptic equation by using the time discretization method, and then the existence and uniqueness solution are obtained. The blow-up results is shown, by using the energy method.

Key words: $p_i(x)$-Laplacian systems, Existence, Uniqueness, Variable exponent, Blow-Up, Semi-discretization.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded Lipshitz domain and $0 < T < \infty$. It will be assumed throughout this paper that $p(x)$ is continuous function defined in $\overline{\Omega}$ with logarithmic module of continuity:

$$\begin{gathered} 2 < p^- = \inf_{\overline{\Omega}} p(x) \leq p(x) \leq p^+ = \sup_{\overline{\Omega}} p(x) < \infty, \\ |p(x) - p(y)| \leq -\frac{C}{\log|x-y|}, \\ \text{for any } x, y \in \Omega \text{ with } |x-y| < \frac{1}{2}. \end{gathered} \tag{1}$$

We set $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. Our aim is to prove the existence and uniqueness of solutions $u = (u_1, u_2)$ to the nonlinear $(p_1(x), p_2(x))$-Laplacian system:

$$\begin{cases} a_1(x)\dfrac{\partial u_1}{\partial t} - \Delta_{p_1(x)}u_1 = f_1(x, u_1, u_2) & \text{in} \quad Q_T, \\ a_2(x)\dfrac{\partial u_2}{\partial t} - \Delta_{p_2(x)}u_2 = f_2(x, u_1, u_2) & \text{in} \quad Q_T, \\ u_1 = u_2 = 0, & \text{in} \quad \Sigma_T, \\ (u_1(.,0), u_2(.,0)) = (\varphi_1, \varphi_2) & \text{on} \quad \Omega. \end{cases} \tag{2}$$

where $p_i(x) \in C(\overline{\Omega})$ is a function, $(i = 1,\, 2)$.

The operator $-\Delta_{p(x)}w = -\operatorname{div}\left(|\nabla w|^{p(x)-2}\nabla w\right)$ is called $p(x)$-Laplacian, which will be reduced to the p-Laplacian when $p(x) = p$ a constant.

The $(p_1(x), p_2(x))$-Laplacian system (2) can be viewed as a generalization of (p, q)-Laplacian system

$$\begin{cases} u_t - \Delta_p u = f(x, u, v) & \text{in} \quad Q_T, \\ v_t - \Delta_q v = g(x, u, v) & \text{in} \quad Q_T, \\ u = v = 0 & \text{in} \quad \Sigma_T, \\ (u(.,0), v(.,0)) = (\varphi_1, \varphi_2) & \text{on} \quad \Omega. \end{cases} \tag{3}$$

For the case $p_i(x) = p_i > 2$, and $a_i(x) = 1$, $(i = 1,\, 2)$, system (2) models as non-Newtonian fluids [2, 21] and nonlinear filtration [2], etc. In the non-Newtonian fluids theory, (p_1, p_2) is a characteristic quantity of the fluids, there have been many results about the existence, uniqueness of the solutions. We refer the readers to the bibliography given in [7, 9, 10, 11, 28, 31] and the references therein.

In recent years, the research of nonlinear problems with variable exponent growth conditions has been an interesting topic. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems and these problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is inhomogeneous. And these problems have many important applications in nonlinear elastic, electrorheological fluids and image restoration. The reader can find in [14, 22] several models in mathematical physics where this class of problem appears.

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The case of a single equation of the type (2) has been studied in [4–6, 20] and the authors established the existence and uniqueness results, in [20], the authors use the difference scheme to transform the parabolic problem to a sequence of elliptic problems and then obtain the existence of solutions with less constraint to $p_i(x)$.

The more interesting question concerning parabolic systems of $(p_1(x), p_2(x))$-Laplacian type is to understand the asymptotic behavior of solutions when time goes to infinity. The study of the asymptotic behavior of the system is giving us relevant information about the structure of the phenomenon described in the model.

Concerning the elliptic systems with variable exponents, the results about existence and non-existence are proved in [26, 27, 29, 30].

Note that system (2) has a more complicated nonlinearity than the classical (p,q)-Laplacian system since it is nonhomogenous.

Recently, [24] study the equation the $p(x)$-Laplacian equation

$$a(x)\frac{\partial u}{\partial t} = \operatorname{div}\left(u^{m-1}|Du|^{\lambda-1}Du\right),$$

where $\lambda > 0$, $m + \lambda - 2 > 0$ and $a(x)$ is a positive continuous function. They examine under which conditions on behavior of $a(x)$ corresponding nonnegative solutions of the Cauchy problems possess the finite speed of propagations or interface blow-up phenomena.

In this paper, we consider the existence and uniqueness for the problem of the type (2) under some assumptions. The proof consists of two steps. First, we prove that the approximating problem admits a global solution; then we do some uniform estimates for these solutions. We mainly use skills of inequality estimation and the method of approximation solutions. By a standard limiting process, we obtain the existence to problem of the type (2).

The outline of this paper is the following: In Section 2, we introduce some basic Lebesgue and Sobolev spaces and state our main theorems. In Section 3, we give the existence and uniqueness of weak solutions. In Section 4, the blow-up results will be proved. The asymptotic behaviour of solution is established in Section 5.

2 Preliminaries

To consider problems with variable exponents, one needs the basic theory of spaces $L^{p(x)}(\Omega)$ and $W^{1p(x)}(\Omega)$. For the convenience of readers, let us review them briefly here. The details and more properties of variable-exponent Lebesgue-Sobolev spaces can be found in [16, 17].

Let $p(x) \in C(\overline{\Omega})$. When $p^- > 1$, one can introduce the variable-exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R};\quad u \text{ is measurable and } \int_\Omega |u|^{p(x)}\mathrm{d}x < \infty\right\},$$

endowed with the Luxemburg norm.

$$||w||_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{w}{\lambda}\right|^{p(x)}\mathrm{d}x \le 1\right\}.$$

The conjugate space is $L^{q(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ $\forall x \in \overline{\Omega}$.

As in the case of a constant exponent, set

$$W^{1,p(x)}(\Omega) = \left\{u(x) \in L^{p(x)}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega)\right\},$$

endowed with the norm

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)}.$$

Similarly we also denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $\left(W_0^{1,p(x)}(\Omega)\right)'$ is the dual of $W_0^{1,p(x)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$.

In Propositions 2.1–2.3, we describe some results about the Luxembourg norm.

Proposition 2.1 [16, 17]

1. The space $\left(L^{p(x)}(\Omega), ||.||_{p(x)}\right)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ $\forall x \in \overline{\Omega}$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality:
$$\left|\int_\Omega uv\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{(q^-)}\right)||u||_{p(x)}||v||_{q(x)} \le 2||u||_{p(x)}||v||_{q(x)}.$$
2. If $r_1(x) \le r_2(x)$ for any $x \in \Omega$, the imbedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$ is continuous, the norm of the imbedding does not exceed $|\Omega| + 1$.

Proposition 2.2 [16]

If we denote

$$\rho(w) = \int_\Omega |w|^{r(x)}\mathrm{d}x, \forall w \in L^{r(x)}(\Omega),$$

then

1. $|w|_{(r(x))} < 1 (= 1; > 1) \iff \rho(w) < 1\ (= 1; > 1)$;
2. $|w|_{r(x)} > 1 \Rightarrow |w|_{r(x)}^{r^-} \le \rho(w) \le |w|_{r(x)}^{r^+}$;
   $|w|_{r(x)} < 1 \Rightarrow |w|_{r(x)}^{r^+} \le \rho(w) \le |w|_{r(x)}^{r^-}$;
3. $|w|_{(r(x))} \to 0 \iff \rho(w) \to 0$;
   $|w|_{(r(x))} \to \infty \iff \rho(w) \to \infty$.

Proposition 2.3 [16]

For $u \in W_0^{1,p(x)}(\Omega)$, there exists a constant $C = C(p, |\Omega|) > 0$, such that:

$$||u||_{p(x)} \le C||\nabla u||_{p(x)},$$

This implies that $||\nabla u||_{p(x)}$ and $||u||_{1,p(x)}$ are equivalent norms of $W_0^{1,p(x)}(\Omega)$.

System (2) does not admit classical solutions in general. So, we introduce weak solutions in the following sence.

Definition 2.4 *A function* $u = (u^1, u^2)$ *is said to be a weak solution of equation* (2)*, if the following conditions are satisfied*:

1. $u_i \in L^\infty(0,T,W_0^{1,p_i(x)}(\Omega)) \cap C(0,T;L^2(\Omega))$, $\frac{\partial u_i}{\partial t} \in L^\infty(0,T,W_0^{-1,p_i'(x)}(\Omega)), (i=1,2)$, such that:
2. For any $\phi_i \in C_0^\infty(Q_T)$
$$\int_0^T \int_\Omega (a_i(x)u_i\phi_{it} - |\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla \phi_i - f_i(x,u_1,u_2)\phi_i)\mathrm{d}x\mathrm{d}t = 0$$
3. $u_i(x,0) = \varphi_i(x)$.

In the study of the global existence of solutions, we need the following hypotheses (H):

(H1) $\varphi_i \in L^\infty(\Omega) \cap W_0^{1,p_i(x)}(\Omega), \ (i=1,2)$,
(H2) $0 < C_i \le a_i(x) \in L^\infty(\Omega), \ (i=1,2)$,
(H3) $f_i(x,u_1,u_2) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}), \ (i=1,2)$.

3 Main results

Remark 3.1 In this paper, we shall denote by c_i, C_i differents constants, depending on $p_i(x)$, T, Ω, but not on n, which may vary from line to line. Sometimes we shall refer to a constant depending on specific parameters $C_i(T)$, etc.

Our main existence result is the following:

Theorem 3.2 *Let (H1)–(H3) hold. Then system (2) admits a unique solution* $u = (u_1,u_2) \in (C((0, T); L^2(\Omega)))^2$. *Moreover, the mapping* $(\varphi_1, \varphi_2) \to (u_1(t), u_2(t))$ *is continuous in* $L^2(\Omega) \times L^2(\Omega)$.

Proof of the main results.

3.1 Existence

We will semi-discrete (2) in time and solve the corresponding elliptic problem. Based on the semi-discrete problem, we construct the corresponding approximate solutions. The key procedure is to establish necessary a priori estimates for finding the limit of the approximate solutions via a compactness argument.

We first consider the discrete scheme (4)

$$\begin{aligned} a_i(x)\frac{u_i^n - u_i^{n-1}}{\tau} - \Delta_{p_i(x)}u_i^n &= f_i(x,u_1^{n-1},u_2^{n-1}) && \text{in } \Omega,\\ u_i^n &= 0 && \text{on } \partial\Omega,\\ u_i^0 &= \varphi_i && \text{in } \Omega, \end{aligned} \tag{4}$$

where $N\tau = T$ and T is a fixed positive real, and $1 \le n \le N$.

Lemma 3.3 *For any fixed n, if* $u_i^{n-1} \in W_0^{1,p_i(x)}(\Omega) \cap \mathbf{L}^\infty(\Omega)$, *Problem* (4) *admits a weak solution* $u_i^n \in W_0^{1,p_i(x)}(\Omega) \cap \mathbf{L}^\infty(\Omega)$.

Proof. On the space $W_0^{1,p_i(x)}(\Omega)$, we consider the functional

$$\Phi(v) = \int_\Omega \frac{1}{p_i(x)}|\nabla v|^{p_i(x)}\mathrm{d}x + \frac{1}{2\tau}\int_\Omega a_i(x)|v|^2\mathrm{d}x - \int_\Omega gv\mathrm{d}x.$$

where $g \in \mathbf{L}^\infty(\Omega)$ is a known function. Using Young's inequality and Proposition 2.1, there exist constants C_1, $C_2 > 0$, such that:

$$\begin{aligned}\Phi(v) &\ge \frac{1}{p_i^+}\int_\Omega |\nabla v|^{p_i(x)}\mathrm{d}x - C_2||g||_{L^2}^2\\ &\ge \frac{1}{p_i^+}||v||_{1,p_1(x)}^{p_1^-} - C_2||g||_{L^2}^2,\end{aligned}$$

hence $\Phi(v) \to \infty$, as $||v||_{1,p_i(x)} \to +\infty$. Since the norm is lower semi-continuous and $\int_\Omega gv\mathrm{d}x$ is continuous functional, $\Phi(v)$ is weakly lower semi-continuous on $W_0^{1,p_i(x)}(\Omega)$ and satisfy the coercive condition. From [14] we conclude that there exists $v^* \in W_0^{1,p_i(x)}(\Omega)$, such that:

$$\Phi(v^*) = \inf_{v \in W_0^{1,p_i(x)}(\Omega)} \Phi(v),$$

and v^* is the weak solutions of the Euler equation corresponding to $\Phi(v)$,

$$a_i(x)\frac{v}{\tau} - \Delta_{p_i(x)}v = g.$$

Choosing $g = f_i(x,u_1^0,u_2^0) + a_i(x)\frac{1}{\tau}u_i^0$, we obtain a weak solution u_i^1 of (4).

$$a_i(x)\frac{u_i^1 - u_i^0}{\tau} - \Delta_{p_i(x)}u_i^1 = f_i(x,u_1^0,u_2^0). \tag{5}$$

Since $|f_i(x,u_1^0,u_2^0)| \le M$, we may prove by induction that (4) has a solution u_i^n in $\mathbf{L}^\infty(\Omega)$. We put $u_i^1 := w_i$ and for any integer $k > 0$, we may take $(w_i - M\tau)_+^k$ as a test function in (5) to get

$$\begin{aligned}&\int_\Omega \frac{1}{\tau}(w_i - M\tau)_+^{k+1}\mathrm{d}x + k\int_\Omega \left|\nabla(w_i - M\tau)_+^{p_i(x)}\right|(w_i - M\tau)_+^{k-1}\mathrm{d}x\\ &= \int_\Omega \frac{1}{\tau}(w_i - M\tau)_+^k u_i^0\mathrm{d}x + \int_\Omega f(x,u_1^0,u_2^0)(w_i - M\tau)_+^k\mathrm{d}x.\end{aligned}$$

By using the Hölder's inequality and $|f_i(x,u_1^0,u_2^0)| \le M$, we have

$$\begin{aligned}\int_\Omega (w_i - M\tau)_+^{k+1}\mathrm{d}x &\le \left(\int_\Omega (w_i - M\tau)_+^{k+1}(u_i^0 + M\tau)\mathrm{d}x\right)\\ &\le \left(\int_\Omega (w_i - M\tau)_+^{k+1}\mathrm{d}x\right)^{\frac{k}{k+1}}\left(\int_\Omega^{k+1} (u_i^0 + M\tau)^{k+1}\mathrm{d}x\right)^{\frac{1}{k+1}}.\end{aligned}$$

We deduce $||(w_i - M\tau)_+||_{L^{k+1}(\Omega)} \le ||u_i^0 + M\tau||_{L^{k+1}(\Omega)}$.

Letting $k \to \infty$, we get $(w_i)_+ \le ||u_i^0||_{L^\infty(\Omega)} + 2M\tau$. Consider $-w_i$, we get easily that $(w_i)_- \ge -||u^0||_{L^\infty(\Omega)} - 2M\tau$, i.e. $||u_i^1||_{L^\infty(\Omega)} \le ||u_i^0||_{L^\infty(\Omega)} + 2M\tau$ and if we choose τ such that $\tau \le \frac{1}{2M}$, we obtain $u_i^n \in \mathbf{L}^\infty(\Omega)$.

This completes the proof of lemma 3.3.

Now, we define the approximate solutions as $(u_i)_\tau$, $(\tilde{u}_i)_\tau$ set by: for all $n \in \{1, \ldots, N\}$.

$$\forall t \in [(n-1)\tau, n\tau] \begin{cases} u_{i\tau}(t) = u_i^n, \\ \tilde{u}_{i\tau}(t) = \frac{(t-(n-1)\tau)}{\tau}\left(u_i^n - u_i^{n-1}\right) + u_i^{n-1}, \end{cases}$$

are well defined and satisfied in addition

$$a_i(x)\frac{\partial \tilde{u}_{i\tau}}{\partial t} - \Delta_{p_{i(x)}} u_{i\tau} = f_i(x, u_{1\tau}(.-\tau), u_{2\tau}(.-\tau)). \quad (6)$$

We first establish some energy estimates of $u_{i\tau}, \tilde{u}_{i\tau}$.

We need several lemmas to complete the proof of Theorem 3.2.

Lemma 3.4 *There exists a positive constant $C(T, u_0)$ such that, for all $n = 1, \ldots, N$*

$$u_i^n \in L^\infty(0, T; L^\infty(\Omega)), \quad (7)$$

$$u_{i\tau}, \tilde{u}_{i\tau} \text{ are bounded in } L^{p_i(x)}\left(0, T; W_0^{1,p_i(x)}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right), \quad (8)$$

$$\frac{\partial \tilde{u}_{i\tau}}{\partial t} \text{ is bounded in } L^2(Q_T), \quad (9)$$

and

$$u_{i\tau}, \tilde{u}_{i\tau} \text{ are bounded in } L^\infty\left(0, T; W_0^{1,p_i(x)}(\Omega)\right). \quad (10)$$

Proof. (a) By lemma 3.3, for any $n \in N$, u_i^n is bounded; whence (7)

(b) Multiplying (4) by τu_i^n, summing from $n = 1$ to N and integrating over Ω, we obtain

$$\tau\sum_{n=1}^N \int_\Omega a_i(x)\left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)u_i^n \mathrm{d}x + \tau\sum_{n=1}^N \int_\Omega \left|\nabla u_i^n\right|^{p_i(x)}\mathrm{d}x = \tau\sum_{n=1}^N \int_\Omega f_i\left(x, u_1^{n-1}, u_2^{n-1}\right)u_i^n \mathrm{d}x. \quad (11)$$

By using the Young's inequality, for $\epsilon > 0$ small, there exists $C_\epsilon(T)$ such that

$$\tau\sum_{n=1}^N \int_\Omega f_i\left(x, u_1^{n-1}, u_2^{n-1}\right)u_i^n \mathrm{d}x \le \epsilon\tau\sum_{n=1}^N \int_\Omega \left|\nabla u_i^n\right|^{p_i(x)}\mathrm{d}x + C_\epsilon(T). \quad (12)$$

With the aid of the identity $2\alpha(\alpha - \beta) = \alpha^2 - \beta^2 + (\alpha - \beta)^2$, we get

$$\begin{aligned}\tau\sum_{n=1}^N \int_\Omega a_i(x)\left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)u_i^n \mathrm{d}x &= \frac{1}{2}\sum_{n=1}^N \int_\Omega a_i(x)\left(\left|u_i^n\right|^2 - \left|u_i^{n-1}\right|^2 + \left|u_i^n - u_i^{n-1}\right|^2\right)\mathrm{d}x \\ &= \frac{1}{2}\sum_{n=1}^N \int_\Omega a_i(x)\left(\left|u_i^n\right|^2 - \left|u_i^{n-1}\right|^2\right)\mathrm{d}x + \frac{1}{2}\int_\Omega a_i(x)\left|u_i^N\right|^2\mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega a_i(x)|\varphi_i|^2\mathrm{d}x.\end{aligned}$$

With the above estimates, we get (8).

(c) Multiplying the equation (4) by $u_i^n - u_i^{n-1}$ and summing from $n = 1$ to N, we get

$$\tau\sum_{n=1}^N \int_\Omega a_i(x)\left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)^2\mathrm{d}x + \sum_{n=1}^N \int_\Omega \left|\nabla u_i^n\right|^{p_i(x)-2}\nabla u_i^n.\nabla\left(u_i^n - u_i^{n-1}\right)\mathrm{d}x = \sum_{n=1}^N \int_\Omega f_i(x, u_1^{n-1}, u_2^{n-1})\left(u_i^n - u_i^{n-1}\right)\mathrm{d}x.$$

By using the Young's inequality, we get

$$\sum_{n=1}^N \int_\Omega f_i\left(x, u_1^{n-1}, u_2^{n-1}\right)\left(u_i^n - u_i^{n-1}\right)\mathrm{d}x \le C_\epsilon(T) + \frac{\tau}{2}\sum_{n=1}^N \int_\Omega \left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)^2\mathrm{d}x. \quad (13)$$

From the convexity of the expression $\int_\Omega |\nabla w|^{p_i(x)}\mathrm{d}x$, we get the following inequality:

$$\int_\Omega \frac{1}{p_i(x)}\left|\nabla u_i^n\right|^{p_i(x)}\mathrm{d}x - \int_\Omega \frac{1}{p_i(x)}\left|\nabla u_i^{n-1}\right|^{p_i(x)}\mathrm{d}x \le \int_\Omega \left|\nabla u_i^n\right|^{p_i(x)-2}\nabla u_i^n.\nabla\left(u_i^n - u_i^{n-1}\right)\mathrm{d}x, \quad (14)$$

which imply with (12) and (13) that

$$\frac{\tau}{2}\sum_{n=1}^N \int_\Omega a_i(x)\left(\frac{u_i^n - u_i^{n-1}}{\tau}\right)^2\mathrm{d}x + \int_\Omega \frac{1}{p_i(x)}\left|\nabla u_i^N\right|^{p_i(x)}\mathrm{d}x \le C_i.$$

By lemma 3.4, there exists $M_i > 0$ independent of τ such that:

$$\|u_{i\tau} - \tilde{u}_{i\tau}\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le \max_{1\le n\le N}\|u_i^n - u_i^{n-1}\|_{L^2(\Omega)} \le M_i\sqrt{\tau}. \quad (15)$$

Therefore, taking $\tau \to 0^+$, and up to subsequence, we get that there exists $u_i, v_i \in L^\infty(0, T; W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega))$ such that $\frac{\partial u_i}{\partial t} \in L^2(Q_T)$, and as $\tau \to 0^+$,

$$u_{i\tau} \overset{*}{\rightharpoonup} u_i \quad \text{in} \quad L^\infty\left(0, T; W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)\right) \quad \text{and}$$
$$\tilde{u}_{i\tau} \overset{*}{\rightharpoonup} v_i \quad \text{in} \quad L^\infty\left(0, T; W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)\right), \quad (16)$$

$$\frac{\partial \tilde{u}_{i\tau}}{\partial t} \rightharpoonup \frac{\partial u_i}{\partial t} \text{ in } L^2(Q_T). \quad (17)$$

From (14), it follows that $u_i = v_i$. From (15), we get that

$$u_{i\tau}, \tilde{u}_{i\tau} \to u_i \quad \text{in} \quad L^\infty(0, T; L^q(\Omega)), \quad \forall q > 1. \quad (18)$$

By Aubin-Simon's compactness results [22], we have

$$\tilde{u}_{i\tau} \to u_i \in C\big(0, T; L^2(\Omega)\big). \tag{19}$$

Now, multiplying (4) by $u_{i\tau} - u_i$ and using (14) and (15), we get by straightforward calculations:

$$\int_0^T \int_\Omega a_i(x)\left(\frac{\partial \tilde{u}_{i\tau}}{\partial t} - \frac{\partial u_i}{\partial t}\right)(\tilde{u}_{i\tau} - u_i)\mathrm{d}x\mathrm{d}t - \int_0^T \langle \Delta_{p_i(x)} u_{i\tau}, u_{i\tau} - u_i\rangle \mathrm{d}t = \int_0^T \int_\Omega f_i(x, u_{1\tau}(. - \tau), u_{2\tau}(. - \tau))\mathrm{d}x\mathrm{d}t + o_\tau(1),$$

where $o_\tau(1) \to 0$ as $\tau \to 0^+$.

Thus, we get that

$$\frac{1}{2}\int_\Omega a_i(x)|\tilde{u}_{i\tau}(T) - u_i(T)|^2 \mathrm{d}x - \int_0^T \langle \Delta_{p_i(x)} u_{i\tau} - \Delta_{p_i(x)} u_i, u_{i\tau} - u_i\rangle \mathrm{d}t \le \int_0^T \int_\Omega f_i(x, u_{1\tau}(. - \tau), u_{2\tau}(. - \tau))\mathrm{d}x\mathrm{d}t + o_\tau(1), \tag{20}$$

and from (16) we have thus,

$$u_{i\tau} \to u_i \quad \text{in} \quad L^{p_i(x)}\left(0, T; W_0^{1,p_i(x)}(\Omega)\right), \qquad \text{as } \tau \to 0^+,$$

and consequently by the same as that in [24]

$$\Delta_{p_i(x)} u_{i\tau} \to \Delta_{p_i(x)} u_i \quad \text{in} \quad L^{p_i(x)'}(0, T; W^{-1,p_i(x)'}(\Omega)).$$

Therefore, u_i satisfies (3).

3.2 Uniqueness

Let (H1)–(H3) be satisfied. Then system (2) has a unique solution $u = (u_1, u_2)$ in Q_T.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be solutions of (2), we have:

$$\int_0^T \int_\Omega a_i(x)\frac{\partial(u_i - v_i)}{\partial t}(u_i - v_i)\mathrm{d}x\mathrm{d}t - \int_0^T \langle \Delta_{p_i(x)} u_i - \Delta_{p_1(x)} v_i, u_i - v_i\rangle \mathrm{d}t = \int_0^T \int_\Omega (f_i(x, u) - f(x, v))(u_i - v_i)\mathrm{d}x\mathrm{d}t.$$

Since $f_i(x,.,.)$ is locally Lipschitz uniformly in Ω, the difference $w_i = u_i - v_i$ satisfies

$$\frac{C}{2}\sum_{i=1}^2 |w_i|^2_{L^2(\Omega)} + \sum_{i=1}^2 \int_0^T \langle \Delta_{p_i(x)} u_i - \Delta_{p_i(x)} v_i, w_i\rangle \mathrm{d}t \le c\sum_{i=1}^2 \int_0^T \int_\Omega |w_i|^2 \mathrm{d}t,$$

we observe that $w \to -\Delta_{p(x)} w$ is monotone from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p(x)'}(\Omega)$

$$\sum_{i=1}^2 |w_i|^2 \le 2c\sum_{i=1}^2 \int_0^T |w_i|^2 \mathrm{d}t. \tag{21}$$

We finally deduce from Gronwall's lemma,

$$\sum_{i=1}^2 |w_i|^2 \le \sum_{i=1}^2 |w_i(0)|^2 \exp(2cT), \quad \forall t \in (0, T).$$

Thus, we deduce that $u_i = v_i$.

Thus the solution is unique. The continuity of the the mapping $(\varphi_1, \varphi_2) \to (u_1(t), u_2(t))$ can be obtained similarly.

Remark 3.5 *From Theorem 3.2, the solution of system* (2) *generates a semigroup* $\{S(t)\}_{t\ge 0}$ *in* $L^2(\Omega) \times L^2(\Omega)$.

Remark 3.6 *If we assume that* $f_i(x, s_1, s_2) = g_i(x, s_1, s_2) - h_i(x, s_i)$ *where* $h_i : \Omega \times \mathbb{R} \to \mathbb{R}$ *is Carathéodory mapping and that there exist positive constants* L_j, c_j, m_j, C_j *such that:*

(i) $(h_i(x, a) - h_i(x, b))(a - b) \ge -L_i|a - b|^2$ for any $x \in \Omega$ and $a, b \in \mathbb{R}, i = (1, 2)$.

(ii) $c_i|s|^{\alpha_i(x)} - m_i \le h_i(x, s)s \le C_i|s|^{\alpha_i(x)} + m_i$ for any $x \in \Omega$ and $a, b \in \mathbb{R}$, where $\alpha_i(x) \in C(\overline{\Omega})$, with $2 \le \alpha_i^- \le \alpha_i^+ < \infty$ and $g_i(x, s_1, s_2) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}), i = (1, 2)$.

where $\alpha_i(x) \in C(\overline{\Omega})$, with $2 \le \alpha_i^- \le \alpha_i^+ < \infty$, $i = (1, 2)$ and $g_i(x, s_1, s_2) \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$.

By the same argument as that in [12, 32], One can show in the same way that the semigroup $\{S(t)\}_{t\ge 0}$ associated with system (2) admits an absorbing set in $\prod_{i=1}^2 \left(W_0^{1,p_i(x)}(\Omega) \cap L^{\alpha_i(x)}(\Omega)\right)$; there is a bounded set $B_0 \subset \prod_{i=1}^2 \left(W_0^{1,p_i(x)}(\Omega) \cap L^{\alpha_i(x)}(\Omega)\right)$ such that, for any bounded set B in $L^2(\Omega) \times L^2(\Omega)$, there exists a $T_0 > 0$ such that $S(t)B \subset B_0$ for any $t \ge T_0$. Where T_0 depends only on B.

4 Blow-up of solutions

In this section, we shall investigate the blow-up properties of solutions to system (2) using energy methods. To this end, we consider the following hypotheses on the data.

(H4) $\varphi_i \in W_0^{1,p(.)}(\Omega) \cap L^{p(.)}(\Omega)$ such that:

$$\int_\Omega F(\varphi_1(x), \varphi_2(x))\mathrm{d}x - \sum_{i=1}^2 \int_\Omega \frac{1}{p_i(x)}|\nabla\varphi_i|^{p_i(x)}\mathrm{d}x \ge 0.$$

(H5) $f_i(x, u_1, u_2) = \dfrac{\partial H}{\partial u_i}(u_1, u_2)$ and H is such that:

$$\sum_{i=1}^2 |u_i|^\alpha \le \alpha H(u_1, u_2) \le \sum_{i=1}^2 u_i\frac{\partial F}{\partial u_i}, \quad \alpha > max(p_2^+, 2).$$

Throughout this section, we define for $t \ge 0$

$$E(t)=\sum_{i=1}^{2}\int_{\Omega}\frac{1}{p_i(x)}|\nabla u_i(x,t)|^{p_i(x)}\mathrm{d}x-\int_{\Omega}H(u_1(x,t),u_2(x,t))\mathrm{d}x$$

Theorem 4.1 *Let (H1)–(H5) be satisfied, then the solutions of system (2) blow up in finite time, namely, there exists a $T^*<\infty$ such that $||u_i(.,t)||_{\infty,\Omega}\to\infty$ as $t\to T^*$.*

Proof. We define $E(t)=\sum_{i=1}^{2}\int_{\Omega}\frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\mathrm{d}x-\int_{\Omega}H(u_1(x,t),u_2(x,t))\mathrm{d}x$.

Multiplying the first equation of (2) by $\frac{\partial u_1}{\partial t}$, the second by $\frac{\partial u_2}{\partial t}$, integrating by parts, we have

$$E'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{i=1}^{2}\int_{\Omega}\frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\mathrm{d}x-\int_{\Omega}H(u_1(x,t),u_2(x,t))\mathrm{d}x\right\}=-\sum_{i=1}^{2}\int_{\Omega}a_i(x)\left(\frac{\partial u_i}{\partial t}\right)^2\mathrm{d}x\leq 0,\tag{22}$$

which implies that $E(t)\leq E(0)$.

Next define $g(t)=\frac{1}{2}\sum_{i=1}^{2}\int_{\Omega}a_i(x)u_i^2\mathrm{d}x$.

Multiplying the first equation of (2) by u_1, the second by u_2, integrating by parts, we have

$$\begin{aligned}g'(t)&=\sum_{i=1}^{2}\int_{\Omega}a_i(x)u_i\frac{\partial u_i}{\partial t}\mathrm{d}x\\&=-\sum_{i=1}^{2}\int_{\Omega}\frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial H}{\partial u_i}\mathrm{d}x\\&=-\sum_{i=1}^{2}\int_{\Omega}p_i(x)\frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial H}{\partial u_i}\mathrm{d}x\\&\geq-\sum_{i=1}^{2}p_i^+\int_{\Omega}\frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial H}{\partial u_i}\mathrm{d}x\\&\geq-p_2^+\left(E(t)+\int_{\Omega}H(u_1(x,t),u_2(x,t))\mathrm{d}x\right)+\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial H}{\partial u_i}\mathrm{d}x\\&\geq\sum_{i=1}^{2}\int_{\Omega}u_i\frac{\partial H}{\partial u_i}\mathrm{d}x-p_2^+\int_{\Omega}H(u_1(x,t),u_2(x,t))\mathrm{d}x\\&\geq\left(\frac{\alpha-p_2^+}{\alpha}\right)\sum_{i=1}^{2}\int_{\Omega}|u_i|^{\alpha}\mathrm{d}x.\end{aligned}\tag{23}$$

By using Hölder's inequality, we have

$$\frac{1}{2}\int_{\Omega}a_i(x)u_i^2\mathrm{d}x\leq c_0\left(\frac{1}{2}\right)|\Omega|^{\frac{\alpha-2}{2}}\left(\int_{\Omega}|u_i|^{\alpha}\mathrm{d}x\right)^{\frac{2}{\alpha}},\tag{24}$$

where $c_0=\max\left(||a_1||_{\infty},||a_2||_{\infty}\right)$.

By the formula $\left(\frac{a+b}{2}\right)^{\beta}\leq a^{\beta}+b^{\beta},\forall a,b>0,\beta>1$, we have by combining (23), (24)

$$g'(t)\geq kg^{\frac{\alpha}{2}}(t),$$

where $k=\left(\frac{1}{c_0}\right)^{\frac{2}{\alpha}}(1-\frac{p_2^+}{\alpha})|\Omega|^{\frac{2-\alpha}{2}}>0$.

A direct integration of the above inequality over (0, t) then yields

$$g^{\frac{\alpha}{2}-1}(t)\geq\frac{1}{g^{1-\frac{\alpha}{2}}(0)-k\left(\frac{\alpha}{2}-1\right)t},$$

which implies that $g(t)$ bows up at a finite time $T^*\leq g^{1-\frac{\alpha}{2}}(0)/\left(k\left(\frac{\alpha}{2}-1\right)\right)$, and so does u^i.

5 Asymptotic behaviour

This section is devoted to the asymptotic behaviour of solutions. In order to prove the asymptotic behaviour, we assume

(H6) $\sum_{i=1}^{2}f_i(x,u_1,u_2)u_i\leq 0$.

Theorem 5.1 *The weak solution $u=(u_1(t),u_2(t))$ obtained in Theorem 3.2, satisfies*: $\int_{\Omega}|u_1(x,t)|^2\mathrm{d}x+\int_{\Omega}|u_2(x,t)|^2\mathrm{d}x\leq\frac{C_1}{(C_2t+C_3)^{\alpha}}$, where $C^i>0$ (i = 1, 2, 3), $\alpha=\frac{2}{\beta-2},\beta=p_1^-$ or p_2^+ or p_2^-.

Proof. Let u_i be solution of (2).

Multiplying the first equation in (2) by u_1 and integrating over Q_T,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}a_1(x)|u_1|^2\mathrm{d}x+\int_{\Omega}|\nabla u_1|^{p_1(x)}\mathrm{d}x=\int_0^T\int_{\Omega}f_1(x,u_1,u_2)u_1\mathrm{d}x.\tag{25}$$

Multiplying the second equation in (2) by u_2 and integrating over Q_T,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}a_2(x)|u_2|^2\mathrm{d}x+\int_{\Omega}|\nabla u_2|^{p_2(x)}\mathrm{d}x=\int_0^T\int_{\Omega}f_2(x,u_1,u_2)u_2\mathrm{d}x.\tag{26}$$

Summing up (25) and (26), we have from hypothesis (H5) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}a_1(x)|u_1|^2\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}a_2(x)|u_2|^2\mathrm{d}x+\int_{\Omega}|\nabla u_1|^{p_1(x)}\mathrm{d}x+\int_{\Omega}|\nabla u_2|^{p_2(x)}\mathrm{d}x\leq 0.\tag{27}$$

By $u_i\in W_0^{1,p_i(x)}(\Omega)$, using Poincaré's inequality, we obtain

$$||u_i||_{L^2}^2\leq c||\nabla u_i||_{L^2}^2\leq c||\nabla u_1||_{p_i(x)}^2.\tag{28}$$

If $|\nabla u_1|_{p_1(x)}>1$ and $|\nabla u_2|_{p_2(x)}>1$, by Proposition 2.2,

$$|\nabla u_1|_{p_1(x)}^{p_1^-} \leq \int_\Omega |\nabla u_1|^{p_1(x)}\mathrm{d}x \quad \text{and}$$

$$|\nabla u_2|_{p_2(x)}^{p_2^-} \leq \int_\Omega |\nabla u_2|^{p_2(x)}\mathrm{d}x. \tag{29}$$

According to the assumption that $p_1(x) \leq p_2(x)$, Then $2 < p_1^- \leq p_1^+ \leq p_2^- \leq p_2^+$.

Hence, we get from (28) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega a_1(x)|u_1|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_2(x)|u_2|^2\mathrm{d}x$$
$$+ C_1\left(\int_\Omega |u_1|^2\mathrm{d}x\right)^{\frac{p_2^-}{2}} + C_2\left(\int_\Omega |u_2|^2\mathrm{d}x\right)^{\frac{p_2^-}{2}} \leq 0, \text{a.e}, t \geq 0. \tag{30}$$

By the formula $\left(\frac{a+b}{2}\right)^\alpha \leq a^\alpha + b^\alpha, \forall a, b > 0, \alpha > 1$, we have

$$\left(\frac{1}{2}\int_\Omega \left[|u_1|^2\mathrm{d}x + |u_2|^2\right]\mathrm{d}x\right)^{\frac{p_2^-}{2}}$$
$$\leq C\left(\int_\Omega |\nabla u_1|^{p_1(x)}\mathrm{d}x\right)^{\frac{p_2^-}{2}} + \left(\int_\Omega |\nabla u_2|^{p_2(x)}\mathrm{d}x\right)^{\frac{p_2^-}{2}}, \tag{31}$$

this implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_1|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_2|^2\mathrm{d}x$$
$$+ C_3\left(\int_\Omega \left[|u_1|^2\mathrm{d}x + |u_2|^2\right]\mathrm{d}x\right)^{\frac{p_2^-}{2}} \leq 0, a.e, t \geq 0, \tag{32}$$

where $C_3 = \min(C_1, C_2)$.

Denote

$$h(t) = \int_\Omega \left[|u_1|^2\mathrm{d}x + |u_2|^2\right]\mathrm{d}x.$$

Then, we obtain from (32) and (H2) that

$$h'(t) + \mathrm{C}h(t)^{\frac{p_2^-}{2}} \leq 0. \tag{33}$$

If $|\nabla u_1|_{p_1(x)} < 1$ and $|\nabla u_2|_{p_2(x)} < 1$, by Proposition 2.2,

$$|\nabla u_1|_{p_1(x)}^{p_1^+} \leq \int_\Omega |\nabla u_1|^{p_1(x)}\mathrm{d}x \quad \text{and} \quad |\nabla u_2|_{p_2(x)}^{p_2^+} \leq \int_\Omega |\nabla u_2|^{p_2(x)},$$

Then we get from (28) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_1(x)|u_1|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_2(x)|u_2|^2\mathrm{d}x$$
$$+ C_1\left(\int_\Omega |u_1|^2\mathrm{d}x\right)^{\frac{p_2^+}{2}} + C_2\left(\int_\Omega |u_2|^2\mathrm{d}x\right)^{\frac{p_2^+}{2}} \leq 0, \text{a.e}, t \geq 0. \tag{34}$$

That is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_1(x)|u_1|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_2(x)|u_2|^2\mathrm{d}x$$
$$+ C_3\left(\int_\Omega \left[|u_1|^2\mathrm{d}x + |u_2|^2\right]\mathrm{d}x\right)^{\frac{p_2^+}{2}} \leq 0, a.e, t \geq 0. \tag{35}$$

Again we have

$$h'(t) + \mathrm{C}h(t)^{\frac{p_2^+}{2}} \leq 0.$$

Similarly, if $|\nabla u_1|_{p_1(x)} > 1$ and $|\nabla u_2|_{p_2(x)} < 1$, or $|\nabla u_1|_{p_1(x)} < 1$ and $|\nabla u_2|_{p_2(x)} > 1$, we can also obtain the similar results

$$h'(t) + \mathrm{C}h(t)^{\frac{p_1^+}{2}} \leq 0, \text{ or } h'(t) + \mathrm{C}h(t)^{\frac{p_2^-}{2}} \leq 0$$

Hence

$$\int_\Omega \left[|u_1|^2\mathrm{d}x + |u_2|^2\right]\mathrm{d}x \leq \frac{C_1}{(C_2 t + C_3)^\alpha},$$

$$\alpha = \frac{2}{\beta - 2}, \beta = p_1^- \text{ or } p_2^+ \text{ or } p_2^-, C_i > 0, \ i = 1, 2, 3.$$

The proof is complete.

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